Methods Review Sheet

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Self-adjoint ODEs
Periodic functions. Fourier series: definition and simple properties; Parseval's theorem. Equations of second order. Self-adjoint differential operators. The Sturm-Liouville equation; eigenfunctions and eigenvalues; reality of eigenvalues and orthogonality of eigenfunctions; eigenfunction expansions (Fourier series as prototype), approximation in mean square, statement of completeness. [5]

PDEs on bounded domains: separation of variables
Physical basis of Laplace's equation, the wave equation and the diffusion equation. General method of separation of variables in Cartesian, cylindrical and spherical coordinates. Legendre's equation: derivation, solutions including explicit forms of P_0, P_1 and P_2, orthogonality. Bessel's equation of integer order as an example of a self-adjoint eigenvalue problem with non-trivial weight.

Examples including potentials on rectangular and circular domains and on a spherical domain (axisymmetric case only), waves on a finite string and heat flow down a semi-infinite rod. [5]

Inhomogeneous ODEs: Green's functions
Properties of the Dirac delta function. Initial value problems and forced problems with two fixed end points; solution using Green's functions. Eigenfunction expansions of the delta function and Green's functions. [4]

Fourier transforms
Fourier transforms: definition and simple properties; inversion and convolution theorems. The discrete Fourier transform. Examples of application to linear systems. Relationship of transfer function to Green's function for initial value problems. [4]

PDEs on unbounded domains
Classification of PDEs in two independent variables. Well posedness. Solution by the method of characteristics. Green's functions for PDEs in 1, 2 and 3 independent variables; fundamental solutions of the wave equation, Laplace's equation and the diffusion equation. The method of images. Application to the forced wave equation, Poisson's equation and forced diffusion equation. Transient solutions of diffusion problems: the error function. [6]

Contents

1 Test Tips

- Make sure you want to take this course for the test. It covers *a lot* of material and you can easily make mistakes. However, this course is very rewarding and much needed for further applied courses in Part II.
- Remember that this course is essentially *tricks* to do questions. The important ones are:
    - Fourier Transforms
    - Green's Function
    - Method of Images
    - Separation of Variables
    - Linear transformation of variables/boundary conditions
- Practice, practice, practice!

2 Fourier Series and Transforms

2.1 Fourier Series

The motivation behind all of this is to find a basis for *functions* so that our lives are easier. Thinking about periodic functions, fourier came up with the idea of $e^{in\theta}$ under the inner product:

$$(f, g) = \int_{-\pi}^{\pi} f^* g \mathrm{d}x$$

Where f^* is the complex conjugate. We can see easily that these functions are orthogonal under this inner product:

$$(e^{im\theta}, e^{in\theta}) = \int_{-\pi}^{\pi} e^{-im\theta} e^{in\theta} \, \mathrm{d}\theta = \int_{-\pi}^{\pi} e^{i(n-m)\theta} \, \mathrm{d}\theta = \begin{cases} 2\pi & n = m \\ 0 & n \neq m \end{cases} = 2\pi\delta_{nm}$$

Now this can be seen as a basis (We haven't normalized this set, but it is ok), where any periodic function defined in $[-2\pi, 2\pi]$ can be expanded as:

$$f(\theta) = \sum_{n \in \mathbb{Z}} \hat{f}_n e^{in\theta},$$

where

$$\hat{f}_n = \frac{1}{2\pi}(e^{in\theta}, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta) \, \mathrm{d}\theta.$$

For real functions, we see that $\hat{f}_n^* = \hat{f}_{-n}$, so we rewrite it in the form:

$$f(\theta) = \hat{f}_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

Where the coefficients are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos n\theta f(\theta) \, \mathrm{d}\theta, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin n\theta f(\theta) \, \mathrm{d}\theta.$$

Note. Wait. What if it doesn't converge? Well, first of all, you are not in Analysis, so we don't care that much. But here are two useful results to know:

- The Fourier series of $f(x)$ would converge to the periodic extension of f if the periodic extension of f is continuous.
- The Fourier series of f converges to the average of the two one-sided limits, $\frac{f(a^-)+f(a^+)}{2}$ at a jump discontinuity at $x = a$.

Now assuming we have lost all our Analysis worries, we have the *"proof"* for the following important theorem:

Theorem (Parseval's theorem)**.**

$$(f, f) = \int_{-\pi}^{\pi} |f(\theta)|^2 \, \mathrm{d}\theta = 2\pi \sum_{n \in \mathbb{Z}} |\hat{f}_n|^2$$

Proof.

$$\begin{aligned}(f,f) &= \int_{-\pi}^{\pi} |f(\theta)|^2 \,\mathrm{d}\theta \\ &= \int_{-\pi}^{\pi} \left(\sum_{m\in\mathbb{Z}} \hat{f}_m^* e^{-im\theta}\right)\left(\sum_{n\in\mathbb{Z}} \hat{f}_n e^{in\theta}\right) \,\mathrm{d}\theta \\ &= \sum_{m,n\in\mathbb{Z}} \hat{f}_m^* \hat{f}_n \int_{-\pi}^{\pi} e^{i(n-m)\theta} \,\mathrm{d}\theta \\ &= 2\pi \sum_{m,n\in\mathbb{Z}} \hat{f}_m^* \hat{f}_n \delta_{mn} = 2\pi \sum_{n\in\mathbb{Z}} |\hat{f}_n|^2\end{aligned}$$

□

2.2 Fourier Transforms

Now what if we like to do Fourier transform on an arbitrary function? Can we still do it? Yes!

Definition (Fourier transform)**.** The *Fourier transform* of a function $f : \mathbb{R} \to \mathbb{C}$ is defined as

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) \,\mathrm{d}x$$

for *all* $k \in \mathbb{R}$. We will also write $\tilde{f}(k) = \mathcal{F}[f(x)]$.

We list some properties. These form the *absolute* core of this section, so remember these! (Proofs are trivial with the exception of the convolution one, which is included)

Linearity $\mathcal{F}[c_1 f(x) + c_2 g(x)] = c_1\mathcal{F}[f(x)] + c_2\mathcal{F}[g(x)]$.

Translation $\mathcal{F}[f(x-a)] = e^{-ika}\mathcal{F}[f(x)]$

Re-phasing $\mathcal{F}[e^{-i\ell x} f(x)] = \tilde{f}(k+\ell)$

Scaling $\mathcal{F}[f(cx)] = \frac{1}{|c|}\tilde{f}\left(\frac{k}{c}\right)$

Differentiation $\mathcal{F}[f'(x)] = ik\mathcal{F}[f(x)]$ and $\mathcal{F}[xf(x)] = i\tilde{f}'(k)$

Convolution For functions $f, g : \mathbb{R} \to \mathbb{C}$, we define the *convolution* as

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y) \,\mathrm{d}y.$$

We then have

$$\begin{aligned}\mathcal{F}[f*g(x)] &= \int_{-\infty}^{\infty} e^{-ikx}\left[\int_{-\infty}^{\infty} f(x-y)g(y) \,\mathrm{d}y\right] \,\mathrm{d}x \\ &= \int_{\mathbb{R}^2} e^{ik(x-y)} f(x-y) e^{-iky} g(y) \,\mathrm{d}y \,\mathrm{d}x \\ &= \int_{\mathbb{R}} e^{-iku} f(u) \,\mathrm{d}u \int_{\mathbb{R}} e^{-iky} g(y) \,\mathrm{d}y \\ &= \mathcal{F}[f]\mathcal{F}[g],\end{aligned}$$

The most important result out of all of these is the differentiation one. it turns differential equations into algebraic equations! So we can solve them trivially:

Example. Suppose we have a differential equation

$$\mathcal{L}(\partial)y = f,$$

where

$$\mathcal{L}(\partial) = \sum_{r=0}^{p} c_r \frac{\mathrm{d}^r}{\mathrm{d}x^r}$$

is a differential operator of pth order with constant coefficients.

Taking the Fourier transform of both sides of the equation, we find

$$\mathcal{F}[\mathcal{L}(\partial)y] = \mathcal{F}[f(x)] = \tilde{f}(k).$$

Now the left hand side magically becomes:

$$c_0\tilde{y}(k) + c_1 ik\tilde{y}(k) + c_2(ik)^2\tilde{y}(k) + \cdots + c_p(ik)^p\tilde{y}(k) = \mathcal{L}(ik)\tilde{y}(k).$$

Where we define $\mathcal{L}(ik)$ as a polynomial in ik. Thus taking the Fourier transform has changed our ordinary differential equation into the algebraic equation

$$\mathcal{L}(ik)\tilde{y}(k) = \tilde{f}(k).$$

Since $\mathcal{L}(ik)$ is just multiplication by a polynomial, we can immediately get

$$\tilde{y}(k) = \frac{\tilde{f}(k)}{\mathcal{L}(ik)}.$$

But we want our solution in the original space. We have:

Theorem (Fourier Inversion Theorem)**.**

$$f(x) = \mathcal{F}^{-1}[f(k)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\tilde{f}(k)\ \mathrm{d}k.$$

Thus, we have $\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}f(x)\ \mathrm{d}x$.

The identity theorem is useful in turning inverse Fourier transforms into Fourier ones. Now we can restate Parseval's Theorem:

Theorem (Parseval's theorem (again))**.** Suppose $f, g : \mathbb{R} \to \mathbb{C}$ are sufficiently well-behaved that $\tilde{f}$ and $\tilde{g}$ exist and we indeed have $\mathcal{F}^{-1}[\tilde{f}] = f, \mathcal{F}^{-1}[\tilde{g}] = g$. Then

$$(f, g) = \int_{\mathbb{R}} f^*(x)g(x)\ \mathrm{d}x = \frac{1}{2\pi}(\tilde{f}, \tilde{g}).$$

In particular, if $f = g$, then

$$\|f\|^2 = \frac{1}{2\pi}\|\tilde{f}\|^2.$$

Proof.

$$\begin{aligned}(f,g) &= \int_{\mathbb{R}} f^*(x)g(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} f^*(x)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\tilde{g}(x)\,\mathrm{d}k\right]\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f^*(x)e^{ikx}\,\mathrm{d}x\right]\tilde{g}(k)\,\mathrm{d}k \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x)e^{-ikx}\,\mathrm{d}x\right]^*\tilde{g}(k)\,\mathrm{d}k \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}^*(k)\tilde{g}(k)\,\mathrm{d}k \\ &= \frac{1}{2\pi}(\tilde{f},\tilde{g}).\end{aligned}$$

□

2.2.1 Fourier Transforms when it Does Not Make Sense

We have

$$\mathcal{F}[\delta(x)] = \int_{-\infty}^{\infty} e^{ikx}\delta(x)\,\mathrm{d}x = 1.$$

Hence we have

$$\mathcal{F}^{-1}[1] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{d}k = \delta(x).$$

Wait. What? No. But as long as we talk about distributions, this *supposedly* makes sense. And someone in functional analysis has definitely came up a way to make this rigorous. So we are fine.

2.3 Amplifiers and Transfer Functions

Suppose we have an amplifier that modifies an input signal $I(t)$ to produce an output $O(t)$. Typically, amplifiers work by modifying the amplitudes and phases of specific frequencies in the output. By Fourier's inversion theorem, we know

$$I(t) = \frac{1}{2\pi}\int e^{i\omega t}\tilde{I}(\omega)\,\mathrm{d}\omega.$$

This $\tilde{I}(\omega)$ is the *resolution* of $I(t)$.

We specify what the amplifier does by the *transfer function* $\tilde{R}(\omega)$ such that the output is given by

$$O(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\tilde{R}(\omega)\tilde{I}(\omega)\,\mathrm{d}\omega.$$

Since this $\tilde{R}(\omega)\tilde{I}(\omega)$ is a product, on computing $O(t) = \mathcal{F}^{-1}[\tilde{R}(\omega)\tilde{I}(\omega)]$, we obtain a *convolution*

$$O(t) = \int_{-\infty}^{\infty} R(t-u)I(u)\,\mathrm{d}u,$$

where

$$R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\tilde{R}(\omega)\,\mathrm{d}\omega$$

is the *response function.* By plugging it directly into the equation above, we see that $R(t)$ is the response to an input signal $I(t) = \delta(t)$.

Assume that we only start providing input at $t = 0$. Then

$$O(t) = \int_{-\infty}^{\infty} R(t-u)I(u)\,\mathrm{d}u = \int_{0}^{t} R(t-u)I(u)\,\mathrm{d}u.$$

As we will see, this is the same form of solution for initial value PDEs using green functions. But first up is discrete Fourier Transform:

2.4 Discrete Fourier Transform

Theory is all good, but reality is otherwise. We almost never have a closed form for f, and integrating to infinity is as bad as it sounds. So first, we limit the integral to $[-R, S]$. Then since we can't really integrate, we numerically add points at:

$$x = x_j = -R + j\frac{R+S}{N}$$

For N a large integer. Then:

$$\tilde{f}(k) \approx \frac{R+S}{N}\sum_{j=0}^{N-1} f(x_j)e^{-ikx_j}.$$

Also, our computer has finite memory, so we can only store finite number of values for $\tilde{f}(k)$. Let's choose these to be at

$$k = k_m = \frac{2\pi m}{R+S}.$$

Then after some cancellation,

$$\begin{aligned}\tilde{f}(k_m) &\approx \frac{R+S}{N}e^{\frac{2\pi imR}{R+S}}\sum_{j=0}^{N-1} f(x_j)e^{\frac{2\pi i}{N}jm} \\ &= (R+S)e^{\frac{2\pi imR}{R+S}}\left[\frac{1}{N}\sum_{j=0}^{N-1} f(x_j)\omega^{-jm}\right] = (R+S)e^{\frac{2\pi imR}{R+S}}F(m)\end{aligned}$$

where

$$\omega = e^{\frac{2\pi i}{N}}$$

is an Nth root of unity.

Now clearly we cannot retrieve the original function from this data set. But we can retrieve the points we used to calculate $\tilde{f}$:

Example. Consider a function $g : G \to \mathbb{C}$ defined by

$$g(\omega^j) = f(x_j).$$

This is actually nothing but just a new notation for $f(x_j)$. Then using this new notation, we have

$$F(m) = \frac{1}{N}\sum_{j=0}^{N-1} \omega^{-jm}g(\omega^j).$$

The space of *all* functions $g : G \to \mathbb{C}$ is a finite-dimensional vector space, isomorphic to $\mathbb{C}^N$. This has an inner product

$$(f, g) = \frac{1}{N} \sum_{j=0}^{N-1} f^*(\omega^j) g(\omega^j).$$

Now let $e_n : G \to \mathbb{C}$ be the function

$$e_m(\omega^j) = \omega^{jm}.$$

Then the set of functions $\{e_m\}$ for $m = 0, \cdots, N-1$ is an orthonormal basis with respect to our inner product. To show this, we can compute

$$(e_m, e_m) = \frac{1}{N} \sum_{j=0}^{N-1} e_m^*(\omega^j) e_m(\omega^j) = \frac{1}{N} \sum_{j=0}^{N-1} \omega^{-jm} \omega^{jm} = \frac{1}{N} \sum_{j=0}^{N-1} 1 = 1.$$

For $n \neq m$, we have

$$(e_n, e_m) = \frac{1}{N} \sum_{j=0}^{N-1} e_n^*(\omega^j) e^m(\omega^j) = \frac{1}{N} \sum_{j=0}^{N-1} \omega^{j(m-n)} = \frac{1}{N} \frac{1 - \omega^{(m-n)N}}{1 - \omega^{m-n}}.$$

Since $m - n$ is an integer and ω is an Nth root of unity, we know that $\omega^{(m-n)N} = 1$. So the numerator is 0. However, since $n \neq m$, $m - n \neq 0$. So the denominator is non-zero. So we get 0. So we can now rewrite our $F(m)$ as

$$F(m) = \frac{1}{N} \sum_{j=0}^{N-1} \omega^{-jm} f(x_j) = \frac{1}{N} \sum_{j=0}^{N-1} e_m^*(\omega^j) g(\omega^j) = (e_m, g).$$

Hence we can expand our g as

$$g = \sum_{m=0}^{N-1} (e_m, g) e_m = \sum_{m=0}^{N-1} F(m) e_m.$$

Writing f instead of g, we recover the formula

$$f(x_j) = g(\omega^j) = \sum_{m=0}^{N-1} F(m) e_m(\omega^j).$$

If we forget about our fs and just look at the g, what we have effectively done is take the Fourier transform of functions taking values on $G = \{1, \omega, \cdots, \omega^{N-1}\} \cong \mathbb{Z}_N$.

Note. If you think about this long enough, this is essentially taking the Fourier transform of g, except now we add everything discretely.

3 Sturm-Liouville Theory

If you do not take Linear Algebra/Analysis II or think you are not comfortable with Linear Algebra concepts, please see Appendix A.

Definition (Adjoint and self-adjoint)**.** The *adjoint* B of a map $A : V \to V$ is a map such that

$$(B\mathbf{u}, \mathbf{v}) = (\mathbf{u}, A\mathbf{v})$$

for all vectors $\mathbf{u}, \mathbf{v} \in V$. A map is then *self-adjoint* if

$$(M\mathbf{u}, \mathbf{v}) = (\mathbf{u}, M\mathbf{v}).$$

One can treat a self-adjoint operator as an infinite-dimensional analog of Hermitian Matrices. It also satisfy the same properties, with the same proofs (cf. Vectors and Matrices):

- All eigenvalues of it are real.
- All eigenvectors corresponding to different eigenvalues are orthogonal

Now we take the vector space of differentiable functions, and our matrices are linear differential operators. We really only care about operators up to degree 2, and we want to know when they are adjoint. We first define the second order differential form in a slightly weird way:

$$\begin{aligned}
\mathcal{L}y &= P\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + R\frac{\mathrm{d}y}{\mathrm{d}x} - Qy \\
&= P\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{R}{P}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{Q}{P}y\right] \\
&= P\left[e^{-\int \frac{R}{P}\,\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\int \frac{R}{P}\,\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x}\right) - \frac{Q}{P}y\right]
\end{aligned}$$

Let $p = \exp\left(\int \frac{R}{P}\,\mathrm{d}x\right)$. Then we can write this as

$$= Pp^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}y}{\mathrm{d}x}\right) - \frac{Q}{P}py\right].$$

We further define $q = \frac{Q}{P}p$. We also drop a factor of Pp^{-1}. Then we are left with

$$\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) - q(x).$$

This is the *Sturm-Liouville form* of the operator. Now let's compute $(f, \mathcal{L}g)$. We integrate by parts numerous times to obtain

$$\begin{aligned}
(f, \mathcal{L}g) &= \int_a^b f^*\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}g}{\mathrm{d}x}\right) - qg\right)\,\mathrm{d}x \\
&= [f^*pg']_a^b - \int_a^b\left(\frac{\mathrm{d}f^*}{\mathrm{d}x}p\frac{\mathrm{d}g}{\mathrm{d}x} + f^*qg\right)\,\mathrm{d}x \\
&= [f^*pg' - f'^*pg]_a^b + \int_a^b\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}f^*}{\mathrm{d}x}\right) - qf^*\right)g\,\mathrm{d}x \\
&= [(f^*g' - f'^*g)p]_a^b + (\mathcal{L}f, g),
\end{aligned}$$

assuming that p, q are real.

So 2nd order linear differential operators are self-adjoint with respect to this norm if p, q are real and the boundary terms vanish. Now we make sure this happens and calls $\mathcal{L}$ a *Sturm-Liouville* Operator (STO). Now to continue developing this, we define:

Definition (Inner product with weight)**.** An *inner product with weight* w, written $(\cdot,\cdot)_w$, is defined by

$$(f,g)_w = \int_a^b f^*(x)g(x)w(x)\ \mathrm{d}x,$$

where w is real, non-negative, and has only finitely many zeroes.

Using this, besides the two inherited properties from self-adjoint operators, a STO has the following critical properties:

Proposition (Proof Non-Examinable)**.** On a *compact* domain, the eigenvalues $\lambda_1, \lambda_2, \cdots$ form a countably infinite sequence and are discrete.

Proposition (Proof Non-Examinable)**.** The eigenfunctions are complete: any function $f : [a,b] \to \mathbb{C}$ (obeying appropriate boundary conditions) can be expanded as

$$f(x) = \sum_n \hat{f}_n y_n(x),$$

where

$$\hat{f}_n = \int y_n^*(x)f(x)w(x)\ \mathrm{d}x.$$

Note (Note to QM Takers)**.** Note how this corresponds exactly with 2 of the axioms in Quantum Mechanics.

Now, unsurprisingly the last result looks a lot like Fourier series, and thus satisfies Parseval's theorem:

Theorem (Parseval's theorem)**.**

$$(f,f)_w = \sum_{n\in\mathbb{Z}} |\hat{f}_n|^2.$$

Proof. We have

$$\begin{aligned}(f,f)_w &= \int_\Omega f^*(x)f(x)w(x)\ \mathrm{d}x \\ &= \sum_{n,m\in\mathbb{Z}} \int_\Omega \hat{f}_n^* y_n^*(x)\hat{f}_m y_m(x)w(x)\ \mathrm{d}x \\ &= \sum_{n,m\in\mathbb{Z}} \hat{f}_n^*\hat{f}_m(y_n,y_m)_w = \sum_{n\in\mathbb{Z}} |\hat{f}_n|^2.\end{aligned}$$

□

3.1 Least squares approximation

Now as above, we can't really add to infinity in real life. So let's say we truncate somewhere and let

$$g(x) = \sum_{k=1}^n c_k y_k(x).$$

We want to minimize w-norm $(f-g, f-g)_w$, the square error between $g(x)$ and $f(x)$. By linearity of the norm, we know that

$$(f-g,f-g)_w = (f,f)_w + (g,g)_w - (f,g)_w - (g,f)_w.$$

To minimize this norm, we want

$$0 = \frac{\partial}{\partial c_j}(f-g, f-g)_w = \frac{\partial}{\partial c_j}[(f,f)_w + (g,g)_w - (f,g)_w - (g,f)_w].$$

We know that the $(f,f)_w$ term vanishes since it does not depend on c_k. Expanding our definitions of g, we can get

$$0 = \frac{\partial}{\partial c_j}\left(\sum_{i=1}^{\infty}|c_k|^n - \sum_{k=1}^{n}\hat{f}_k^* c_k - \sum_{k=1}^{n} c_k^* \hat{f}_k\right) = c_j^* - \hat{f}_j^*.$$

Note that here we are treating c_j^* and c_j as distinct quantities. So when we vary c_j, c_j^* is unchanged. To formally justify this treatment, we can vary the real and imaginary parts separately.

Hence, the extremum is achieved at $c_j^* = \hat{f}_j^*$. Similarly, varying with respect to c_j^*, we get that $c_j = \hat{f}_j$.

To check that this is indeed an minimum, we can look at the second-derivatives

$$\frac{\partial^2}{\partial c_i \partial c_j}(f-g, f-g)_w = \frac{\partial^2}{\partial c_i^* c_j^*}(f-g, f-g)_w = 0,$$

while

$$\frac{\partial^2}{\partial c_i^* \partial c_j}(f-g, f-g)_w = \delta_{ij} \geq 0.$$

Hence this is indeed a minimum.

Thus we know that $(f-g, f-g)_w$ is minimized over all $g(x)$ when

$$c_k = \hat{f}_k = (y_k, f)_w.$$

These are exactly the coefficients in our infinite expansion. Hence if we truncate our series at an arbitrary point, it is the best approximation using these many eigenfunctions.

4 Partial Differential Equations

In this section, tricks to solving differential equations will be explored.

4.1 Laplace's Equation

This, material wise, is the center of this course, although personally I would like to argue Green's functions are the way forward. But do be *very* comfortable with Laplace's equations, in whatever form they appear:

Definition (Laplace's equation)**.** *Laplace's equation* on $\mathbb{R}^n$ says that a (twice-differentiable) equation $\phi : \mathbb{R}^n \to \mathbb{C}$ obeys

$$\nabla^2 \phi = 0,$$

where

$$\nabla^2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Functions that obey this equation are called harmonic functions.

Proposition. Let Ω be a compact domain, and $\partial\Omega$ be its boundary. Let $f : \partial\Omega \to \mathbb{R}$. Then there is a unique solution to $\nabla^2\phi = 0$ on Ω with $\phi|_{\partial\Omega} = f$.

Proof. Suppose ϕ_1 and ϕ_2 are both solutions such that $\phi_1|_{\partial\Omega} = \phi_2|_{\partial\Omega} = f$. Then let $\Phi = \phi_1 - \phi_2$. So $\Phi = 0$ on the boundary. So we have

$$0 = \int_\Omega \Phi \nabla^2 \Phi \, \mathrm{d}^n x = -\int_\Omega (\nabla\Phi)\cdot(\nabla\Phi)\, \mathrm{d}x + \int_{\partial\Omega} \Phi \nabla\Phi \cdot \mathbf{n} \, \mathrm{d}^{n-1}x.$$

We note that the second term vanishes since $\Phi = 0$ on the boundary. So we have

$$0 = -\int_\Omega (\nabla\Phi)\cdot(\nabla\Phi)\, \mathrm{d}x.$$

However, we know that $(\nabla\Phi)\cdot(\nabla\Phi) \geq 0$ with equality iff $\nabla\Phi = 0$. Hence Φ is constant throughout Ω. Since at the boundary, $\Phi = 0$, we have $\Phi = 0$ throughout, ie. $\phi_1 = \phi_2$. □

4.1.1 Laplace in Unit Disk in $\mathbb{R}^2$

If we define $z = x + iy$ and $\bar{z} = z - iy$, then we have:

$$0 = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\phi}{\partial z \partial \bar{z}},$$

The general solution is $\phi(z, \bar{z}) = \psi(z) + \chi(\bar{z})$ for some ψ, χ. If we want to satisfy $\phi(z, \bar{z}) = f(\theta)$ on the boundary, then we write:

$$f(\theta) = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{in\theta} = \hat{f}_0 + \sum_{n=1}^{\infty} \hat{f}_n e^{in\theta} + \sum_{n=1}^{\infty} \hat{f}_{-n} e^{-in\theta}.$$

By Fourier expansion. On the boundary we then have $z = e^{i\theta}$, so we define:

$$\phi(z, \bar{z}) = \hat{f}_0 + \sum_{n=1}^{\infty} \hat{f}_n z^n + \sum_{n=1}^{\infty} \hat{f}_{-n} \bar{z}^n.$$

This is a solution, and by the proposition above it is unique!

4.1.2 Separation of Variables

Now we enter the world of guessing solutions. Assume we want to solve $|nabla^2\psi = 0$ in 3D with the following boundary conditions:

$$\begin{aligned}\psi(0,y,z) = \psi(a,y,z) &= 0\\ \psi(x,0,z) = \psi(x,b,z) &= 0\\ \lim_{z\to\infty}\psi(x,y,z) &= 0\\ \psi(x,y,0) &= f(x,y),\end{aligned}$$

We look for a form $\psi(x,y,z) = X(x)Y(y)Z(z)$. We expand and reach that:

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0.$$

Since this is true for all x, y, z, if we only change x the two other terms do not change, so the x term cannot change either. Thus, we have:

$$X'' = -\lambda X, \quad Y'' = -\mu Y, \quad Z'' = (\lambda+\mu)Z.$$

(The signs are from hindsight) giving

$$\begin{aligned}X &= a\sin\sqrt{\lambda}x + b\cos\sqrt{\lambda}x,\\ Y &= c\sin\sqrt{\lambda}y + d\cos\sqrt{\lambda}x,\\ Z &= g\exp(\sqrt{\lambda+\mu}z) + h\exp(-\sqrt{\lambda+\mu}z).\end{aligned}$$

The boundary conditions then give

$$\psi(x,y,z) = A_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\exp(-s_{n,m}z),$$

where $A_{n,m}$ is an arbitrary constant, and

$$s_{n,m}^2 = \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2.$$

Then by linearity, any linear combination of it is fine. Now we need to find $A_{m,n}$ to satisfy the inhomogeneous boundary condition at $z = 0$. We use the orthogonality of $sinx$:

$$\sum_{n,m=1}^{\infty} A_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) = f(x,y)$$

Gives

$$\sum_{n,m=1}^{\infty} A_{n,m}\int_0^a \sin\left(\frac{k\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\,\mathrm{d}x\sin\left(\frac{m\pi y}{b}\right) = \int_0^a \sin\left(\frac{k\pi x}{a}\right)f(x,y)\,\mathrm{d}x.$$

Using the orthogonality relation, we have

$$\frac{a}{2}\sum_{m=1}^{\infty} A_{k,m}\sin\left(\frac{m\pi y}{b}\right) = \int_0^a \sin\left(\frac{k\pi x}{a}\right)f(x,y)\,\mathrm{d}x.$$

We do this again to find

$$\frac{ab}{4}A_{k,j} = \int_{[0,a]\times[0,b]} \sin\left(\frac{k\pi x}{a}\right)\sin\left(\frac{j\pi y}{b}\right)f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

And thus we are done.

4.1.3 Spherical Polar Coordinates

The Laplacian in spherical coordinates is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

We look for axisymmetric solutions where $\psi(r,\theta,\phi) = R(r)\Theta(\theta)$. Then similarly we grind through some algebra to find:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = \lambda R, \quad \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) = -\lambda\sin\theta\Theta.$$

4.1.3.1 Angular Equation

For the Angular equation, we let $x = \cos\theta$, and the equation becomes:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right] = -\lambda\Theta.$$

After some chain rule applications. This operator is a Sturm-Liouville operator with

$$p(x) = 1 - x^2, q(x) = 0.$$

For the Sturm-Liouville operator to be self-adjoint, we had

$$(g, \mathcal{L}f) = (\mathcal{L}g, f) + [p(x)(g^* f' - g'^* f)]_{-1}^{1}.$$

We want the boundary term to vanish. Since $p(x) = 1-x^2$ vanishes at our boundary $x = \pm 1$, the Sturm-Liouville operator is self-adjoint provided our function $\Theta(x)$ is regular at $x = \pm 1$. Hence we look for a set of Legendre's equations inside $(-1, 1)$ that remains regular including at $x = \pm 1$. We can try a power series

$$\Theta(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Using techniques from IA Differential Equations:

$$a_{n+2} = \frac{n(n+1)-\lambda}{(n+2)(n+1)}a_n.$$

This relates a_{n+2} to a_n so we have 2 linearly independent solutions. We expand the recurrence formula to obtain:

$$\Theta_0(x) = a_0\left[1 - \frac{\lambda}{2}x^2 - \frac{(6-\lambda)\lambda}{4!}x^4 + \cdots\right]$$
$$\Theta_1(x) = a_1\left[x + \frac{(2-\lambda)}{3!}x^2 + \frac{(12-\lambda)(2-\lambda)}{5!}x^5 + \cdots\right].$$

However, we can show (no we can't but someone can) that is not finite at $x = -1$ or 1. Therefore, for this to work, we need to let the series terminate, so λ can only

take some specific values, namely $\lambda = l(l+1)$, for $l \in \mathbb{N}$. The resulting functions are polynomials:

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x),$$

Or in general:

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}(x^2-1)^\ell.$$

Now we would show explicitly the orthogonality of these functions, with weight function 1. We use the following lemma:

Lemma. For $0 \le k \le \ell$, we have

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^2-1)^\ell = Q_{\ell,k}(x^2-1)^{\ell-k}$$

for some degree k polynomial $Q_{\ell,k}(x)$.

The proof of this is quickly done by induction. Then we have:

$$\begin{aligned}
(P_\ell, P_{\ell'}) &= \int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\,\mathrm{d}x \\
&= \frac{1}{2^\ell \ell!}\int_{-1}^{1}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}(x^2-1)^\ell P_{\ell'}(x)\,\mathrm{d}x \\
&= \frac{1}{2^\ell \ell!}\left[\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}x^{\ell-1}}(x^2-1)^l P_{\ell'}(x)\right] - \frac{1}{2^\ell \ell!}\int_{-1}^{1}\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}x^{\ell-1}}(x^2-1)^\ell \frac{\mathrm{d}P_{\ell'}}{\mathrm{d}x}\,\mathrm{d}x \\
&= -\frac{1}{2^\ell \ell!}\int_{-1}^{1}\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}x^{\ell-1}}(x^2-1)^\ell \frac{\mathrm{d}P_{\ell'}}{\mathrm{d}x}\,\mathrm{d}x.
\end{aligned}$$

Now wlog $l' < l$. Then we integrate by parts l' times so that the second term in the expression becomes 0. So they are orthogonal.

Now we prove that all of its roots are real:

Lemma. For $P_l(x)$, all of its roots are real and lie in $(-1, 1)$.

Suppose not. Suppose only $m < \ell$ roots lie in $(-1, 1)$. Then let $Q_m(x) = \prod_{r=1}^{m}(x - x_r)$, where $\{x_1, x_2, \cdots, x_m\}$ are these m roots. Consider the polynomial $P_\ell(x)Q_m(x)$. If we factorize this, we get $\prod_{r=m+1}^{\ell}(x-r)\prod_{r=1}^{m}(x-x_r)^2$. The first few terms have roots outside $(-1, 1)$, and hence do not change sign in $(-1, 1)$. The latter terms are always non-negative. Hence for some appropriate sign, we have

$$\pm\int_{-1}^{1} P_\ell(x)Q_m(x)\,\mathrm{d}x > 0.$$

However, we can expand

$$Q_m(x) = \sum_{r=1}^{m} q_r P_r(x),$$

but $(P_\ell, P_r) = 0$ for all $r < \ell$. This is a contradiction.

4.1.3.2 Radial Equation

Now setting $\Theta(\theta) = P_l(\cos\theta)$, the radial equation becomes:

$$(r^2 R')' = \ell(\ell+1)R.$$

We now try $R(r) = r^\alpha$, so $\alpha = l$ or $-(l+1)$. Then we have:

$$\phi(r,\theta) = \sum \left(a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Is the general solution, in which we can then use to satisfy specific boundary conditions.

4.1.3.3 Multipole Expansions for Laplace's Equation

We can check, or use the result from EM that

$$\phi(\mathbf{r}) = \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

solves Laplace's equation $\nabla^2\phi = 0$ for all $\mathbf{r} \in \mathbb{R}^3 \setminus \mathbf{r}'$. For example, if $\mathbf{r}' = \hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ is the unit vector in the z direction, then

$$\frac{1}{|\mathbf{r} - \hat{\mathbf{k}}|} = \frac{1}{\sqrt{r^2 + 1 - 2r\cos\theta}} = \sum_{\ell=0}^{\infty} c_\ell r^\ell P_\ell(\cos\theta).$$

To find these coefficients, we can employ a little trick. Since $P_\ell(1) = 0$, at $\theta = 0$, we have

$$\sum_{\ell=0}^{\infty} c_\ell r^\ell = \frac{1}{1-r} = \sum_{\ell=0}^{\infty} r^\ell.$$

So all the coefficients must be 1. This is valid for $r < 1$. More generally, we have

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r'} \sum_{\ell=0}^{\infty} \left(\frac{r}{r'} \right)^\ell P_\ell(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}').$$

This is called the *multiple expansion*, and is valid when $r < r'$. Thus

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r'} + \frac{r}{r'^2} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' + \cdots.$$

The first term $\frac{1}{r'}$ is known as the *monopole*, and the second term is known as the *dipole*, in analogy to charges in electromagnetism. The monopole term is what we get due to a single charge, and the second term is the result of the interaction of two charges.

4.1.4 Laplace's Equation in Cylindrical Coordinates

Again? Seriously?

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0.$$

With Boundary conditions

$$\begin{aligned}\phi(a,\theta,z) &= 0\\ \phi(r,\theta,0) &= f(r,\theta)\\ \lim_{z\to\infty}\phi(r,\theta,z) &= 0,\end{aligned}$$

Thus we get that:

$$\frac{r}{R}(rR')' + \frac{\Theta''}{\Theta} + \mu r^2 = 0.$$

By separation of variables r, θ and z. Now we have:

$$Z'' = \mu Z \qquad \Theta'' = -\lambda\Theta$$

So we are left with:

$$r^2R'' + rR' + (\mu r^2 - \lambda)R = 0.$$

Now we introduce $x = r\sqrt{\mu}$ and rewrite this as:

$$x^2\frac{\mathrm{d}^2R}{\mathrm{d}x^2} + x\frac{\mathrm{d}R}{\mathrm{d}x} + (x^2 - n^2)R = 0.$$

Bessel's Equations give 2 second-order solutions, called (J_n) and (Y_n), the first and second kind. We will just note the properties:

- (J_n) are all regular at origin, and as $x \to 0$, $J_n(x) \approx x^n$.
- (Y_n) are singular at origin, as $x \to 0$, we have $Y_0(x) \approx \ln(x)$ and $Y_n(x) \approx x^{-n}$. Then our general solution is:

  $$\phi(r,\theta,z) = (a_n \sin n\theta + b_n \cos n\theta)e^{-\sqrt{\mu}z}[c_{\mu,n}J_n(r\sqrt{\mu}) + d_{\mu,n}Y_n(r,\sqrt{\mu})].$$

  Where we assumed that we need to decay in the z direction.

4.2 Heat Equation

Definition (Heat equation)**.** The *heat equation* for a function $\phi : \Omega \to \mathbb{C}$ is

$$\frac{\partial\phi}{\partial t} = \kappa\nabla^2\phi,$$

where $\kappa > 0$ is the *diffusion constant*.

It is also called a diffusion equation as it spreads and smooth the function out over time, suitable for heat study. Now we have two useful properties:

- Total *"heat"* is conserved:

  $$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\phi(\mathbf{x},t)\,\mathrm{d}^n x = \int_{\mathbb{R}^n}\frac{\partial\phi}{\partial t}\,\mathrm{d}^n x = \kappa\int_{\mathbb{R}^n}\nabla^2\phi\,\mathrm{d}^n x = 0,$$

- If $\phi(\mathbf{x},t)$ solves the heat equation, so do $\phi_1(\mathbf{x},t) = \phi(\mathbf{x}-\mathbf{x}_0, t-t_0)$, and $\phi_2(\mathbf{x},t) = A\phi(\lambda\mathbf{x},\lambda^2 t)$.

We have solved this before in IA Differential Equations to give:

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right).$$

Now generally on $\mathbb{R}^n$, we have:

$$\phi(\mathbf{x},t) = \frac{1}{(4\pi(t-t_0))^{n/2}} \exp\left(\frac{-(\mathbf{x}-\mathbf{x}_0)^2}{4\kappa(t-t_0)}\right).$$

Note. This is *NOT* the general form for all solutions. This is just one particular class of functions. If you need to solve the heat equation on the test, use separation of variables $\phi(x,t) = X(x)T(t)$ similar to the examples above.

Proposition. Suppose $\phi : \Omega \times [0,\infty) \to \mathbb{R}$ satisfies the heat equation $\frac{\partial \phi}{\partial t} = \kappa \nabla^2 \phi$, and obeys

- Initial conditions $\phi(\mathbf{x}, 0) = f(x)$ for all $x \in \Omega$
- Boundary condition $\phi(\mathbf{x},t)|_{\partial\Omega} = g(\mathbf{x},t)$ for all $t \in [0,\infty)$.

Then $\phi(\mathbf{x},t)$ is unique.

Proof. Suppose ϕ_1 and ϕ_2 are both solutions. Then define $\Phi = \phi_1 - \phi_2$ and

$$E(t) = \frac{1}{2}\int_\Omega \Phi^2 \, \mathrm{d}V.$$

Then we know that $E(t) \geq 0$. Since ϕ_1, ϕ_2 both obey the heat equation, so does Φ. Therefore, on the boundary and at $t = 0$, we know that $\Phi = 0$. We have

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \int_\Omega \Phi \frac{\mathrm{d}\Phi}{\mathrm{d}t} \, \mathrm{d}V \\ &= \kappa \int_\Omega \Phi \nabla^2 \Phi \, \mathrm{d}V\end{aligned}$$

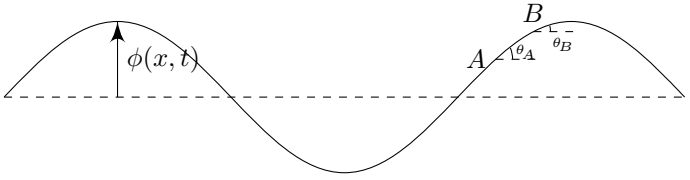
Now we use integration by parts in 3D

$$= \kappa \int_{\partial\Omega} \Phi \nabla\Phi \cdot \mathrm{d}\mathbf{S} - \kappa \int_\Omega (\nabla\Phi)^2 \, \mathrm{d}V = -\kappa \int_\Omega (\nabla\Phi)^2 \, \mathrm{d}V \leq 0.$$

So $E = 0$, nonnegative, and decreasing. So $E = 0$ for all t. So we are done. □

4.3 Wave Equation

Consider a string $x \in [0, L]$ undergoing small oscillations described by $\phi(x,t)$.



Consider two points A, B separated by a small distance δx. Let T_A (T_B) be the outward pointing tension tangent to the string at A (B). Since there is no sideways (x) motion, there is no net horizontal force. So

$$T_A \cos\theta_A = T_B \cos\theta_B = T. \qquad (*)$$

If the string has mass per unit length μ, then in the vertical direction, Newton's second law gives

$$\mu \delta x \frac{\partial^2 \phi}{\partial t^2} = T_B \sin\theta_B - T_A \sin\theta_A.$$

We now divide everything by T, noting the relation $(*)$, and get

$$\begin{aligned} \mu \frac{\delta x}{T} \frac{\partial^2 \phi}{\partial t^2} &= \frac{T_B \sin\theta_B}{T_B \cos\theta_B} - \frac{T_A \sin\theta_A}{T_A \cos\theta_A} \\ &= \tan\theta_B - \tan\theta_A \\ &= \left.\frac{\partial \phi}{\partial x}\right|_B - \left.\frac{\partial \phi}{\partial x}\right|_A \approx \delta x \frac{\partial^2 \phi}{\partial x^2}. \end{aligned}$$

Taking the limit $\delta x \to 0$ and setting $c^2 = T/\mu$, we have that $\phi(x,t)$ obeys the wave equation

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2}.$$

Now we have solved this in one dimension in IA but we are going to this now through separation of variables.

Assume that the string is fixed at both ends. Then $\phi(0,t) = \phi(L,t) = 0$ for all t. The we can perform separation of variables, and the general solution can then be written

$$\phi(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right)\right].$$

The coefficients A_n are fixed by the initial profile $\phi(x,0)$ of the string, while the coefficients B_n are fixed by the initial string velocity $\frac{\partial \phi}{\partial t}(x,0)$. Note that we need *two* sets of initial conditions, since the wave equation is second-order in time.

4.4 Energy and Uniqueness

Now quoting from the integral of Hooke's law and the common expression $\frac{1}{2}mv^2$, we can derive the expressions for potential and kinetic energy respectively as:

$$K(t) = \frac{1}{2}\mu \delta x \left(\frac{\partial \phi}{\partial t}\right)^2 \qquad V(t) = \frac{\mu}{2}\int_0^L c^2 \left(\frac{\partial \phi}{\partial x}\right)^2 \, \mathrm{d}x$$

By evaluating the expressions using the form we have, we can show that averaging over a period, kinetic and potential energy each takes up half of the energy. But we will not do that. Instead, we would prove that energy is conserved:

Proposition. Suppose $\phi : \Omega \times [0,\infty) \to \mathbb{R}$ obeys the wave equation $\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi$ inside $\Omega \times (0,\infty)$, and is fixed at the boundary. Then E is constant.

Proof. By definition, we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_\Omega \frac{\partial^2 \psi}{\partial t^2}\frac{\partial \psi}{\partial t} + c^2 \nabla\left(\frac{\partial \phi}{\partial t}\right)\cdot \nabla\phi \,\mathrm{d}V.$$

We integrate by parts in the second term to obtain

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_\Omega \frac{\mathrm{d}\phi}{\mathrm{d}t}\left(\frac{\partial^2 \phi}{\partial t^2} - c^2\nabla^2\phi\right)\,\mathrm{d}V + c^2\int_{\partial\Omega}\frac{\partial\phi}{\partial t}\nabla\phi\cdot \mathrm{d}S.$$

Since $\frac{\partial^2\phi}{\partial t^2} = c^2\nabla^2\phi$ by the wave equation, and ϕ is constant on $\partial\Omega$, we know that

$$\frac{\mathrm{d}E_\phi}{\mathrm{d}t} = 0.$$

□

Proposition. Suppose $\phi : \Omega \times [0,\infty) \to \mathbb{R}$ obeys the wave equation $\frac{\partial^2\phi}{\partial t^2} = c^2\nabla^2\phi$ inside $\Omega \times (0,\infty)$, and obeys, for some f, g, h,

(i) $\phi(x, 0) = f(x)$;

(ii) $\frac{\partial\phi}{\partial t}(x, 0) = g(x)$; and

(iii) $\phi|_{\partial\Omega\times[0,\infty)} = h(x)$.

Then ϕ is unique.

Proof. Suppose ϕ_1 and ϕ_2 are two such solutions. Then $\psi = \phi_1 - \phi_2$ obeys the wave equation

$$\frac{\partial^2\psi}{\partial t^2} = c^2\nabla^2\psi,$$

and

$$\psi|_{\partial\Omega\times[0,\infty)} = \psi|_{\Omega\times\{0\}} = \left.\frac{\partial\psi}{\partial t}\right|_{\Omega\times\{0\}} = 0.$$

We let

$$E_\psi(t) = \frac{1}{2}\int_\Omega \left[\left(\frac{\partial\psi}{\partial t}\right)^2 + c^2\nabla\psi\cdot\nabla\psi\right]\,\mathrm{d}V.$$

Then since ψ obeys the wave equation with fixed boundary conditions, we know E_ψ is constant.

Initially, at $t = 0$, we know that $\psi = \frac{\partial\psi}{\partial t} = 0$. So $E_\psi(0) = 0$. At time t, we have

$$E_\psi = \frac{1}{2}\int_\Omega \left(\frac{\partial\psi}{\partial t}\right)^2 + c^2(\nabla\psi)\cdot(\nabla\psi)\,\mathrm{d}V = 0.$$

Hence we must have $\frac{\partial\psi}{\partial t} = 0$. So ψ is constant. Since it is 0 at the beginning, it is always 0. □

4.5 Green's Functions, Distributions and Applications

4.5.1 Distributions

No, this is not Statistics. We define a distribution because sometimes we need some functions that are not actually functions, and we would like to do some stuff with their derivatives. To illustrate the definition of a distribution, first pick some set of nice test functions $\phi(x)$. This functions should be bounded and has compact support (0 outside a compact set). Now we create a linear map from the function to the real line, namely the dual space. we call this a *distribution*.

Example. The simplest example is just an ordinary function that is integrable over any compact region. Then we define the distribution T_f as

$$T_f[\phi] = \int_\Omega f(x)\phi(x)\ \mathrm{d}x.$$

However, the only example that we actually care is the Dirac Delta *distribution*:

$$\delta[\phi] = \phi(0) = \int_\Omega \delta(x)\phi(x)\ \mathrm{d}x.$$

The second expression is really an abuse of notation, but well who cares. Now as we said above, a very good thing about distributions is that all their derivatives are defined:

$$T'[\phi] = -T[\phi'].$$

Through integration by parts. Now we thus have the following properties for $\delta(x)$:

Translation $\int_{-\infty}^{\infty} \delta(x-a)\phi(x)\ \mathrm{d}x = \int_{-\infty}^{\infty} \delta(y)\phi(y+a)\ \mathrm{d}x. = \phi(a)$

Scaling $\int_{-\infty}^{\infty} \delta(cx)\phi(x)\ \mathrm{d}x = \frac{1}{|c|}\phi(0)$

General For $f(x)$ a C^1 function with isolated *simple* zeros at x_i, we have:

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(f(x))\phi(x)\ \mathrm{d}x &= \sum_{i=1}^{n} \int_{-\infty}^{\infty} \delta\left((x-x_i)\left.\frac{\partial f}{\partial x}\right|_{x_i}\right)\phi(x)\ \mathrm{d}x \\ &= \sum_{i=1}^{n} \frac{1}{|f'(x_i)|}\phi(x_i).\end{aligned}$$

4.5.2 Green's Functions

This is really the heart of the course as it has many lasting implications for future courses. Let's try to solve $\mathcal{L}y = f(x)$ with $\mathcal{L}$ a differential operator:

$$\mathcal{L} = \alpha(x)\frac{\partial^2}{\partial x^2} + \beta(x)\frac{\partial}{\partial x} + \gamma(x).$$

Now we find Green's function for $\mathcal{L}$ which obeys:

$$\mathcal{L}G(x,\xi) = \delta(x-\xi).$$

Given $G(x,\xi)$, we can then define

$$y(x) = \int_a^b G(x,\xi)f(\xi)\ \mathrm{d}\xi.$$

Then we have

$$\mathcal{L}y = \int_a^b \mathcal{L}G(x,\xi)f(\xi)\,\mathrm{d}\xi = \int_a^b \delta(x-\xi)f(\xi)\,d\xi = f(x).$$

Now we want to find $G(x,\xi)$ obeying $G(a,\xi) = G(b,\xi) = 0$

first note that $\mathcal{L}G = 0$ for all $x \in [a,\xi) \cup (\xi, b]$, ie. everywhere except ξ itself. Thus we must be able to expand it in a basis of solutions in these two regions.

Suppose $\{y_1(x), y_2(x)\}$ are a basis of solutions to $\mathcal{L}G = 0$ everywhere on $[a,b]$, with boundary conditions $y_1(a) = 0, y_2(b) = 0$. Then we must have

$$G(x,\xi) = \begin{cases} A(\xi)y_1(x) & a \le x < \xi \\ B(\xi)y_2(x) & \xi < x \le b \end{cases}$$

So we have a whole family of solutions. To fix the coefficients, we must decide how to join these solutions together over $x = \xi$.

If $G(x,\xi)$ were discontinuous at $x = \xi$, then $\partial_x G|_{x=\xi}$ would involve a δ function, while $\partial_x^2 G|_{x=\xi}$ would involve the derivative of the δ function. This is not good, since nothing in $\mathcal{L}G = \delta(x-\xi)$ would balance a δ'. So $G(x,\xi)$ must be everywhere continuous. Hence we require

$$A(\xi)y_1(\xi) = B(\xi)y_2(\xi). \tag{$*$}$$

Now integrate over a small region $(\xi-\varepsilon, \xi+\varepsilon)$ surrounding ξ. Then we have

$$\int_{\xi-\varepsilon}^{\xi+\varepsilon} \left[\alpha(x)\frac{\mathrm{d}^2 G}{\mathrm{d}x^2} + \beta(x)\frac{\mathrm{d}G}{\mathrm{d}x} + \gamma(x)G\right]\,\mathrm{d}x = \int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x-\xi)\,\mathrm{d}x = 1.$$

By continuity of G, we know that the γG term does not contribute. While G' is discontinuous, it is still finite. So the $\beta G'$ term also does not contribute. So we have

$$\lim_{\varepsilon\to 0}\int_{\xi-\varepsilon}^{\xi+\varepsilon} \alpha G''\,\mathrm{d}x = 1.$$

We now integrate by parts to obtain

$$\lim_{\varepsilon\to 0}[\alpha G']_{\xi-\varepsilon}^{\xi+\varepsilon} + \int_{\xi-\varepsilon}^{\xi+\varepsilon} \alpha' G'\,\mathrm{d}x = 1.$$

Again, by finiteness of G', the integral does not contribute. So we know that

$$\alpha(\xi)\left(\left.\frac{\partial G}{\partial x}\right|_{\xi^+} - \left.\frac{\partial G}{\partial x}\right|_{\xi^-}\right) = 1$$

Hence we obtain

$$B(\xi)y_2'(\xi) - A(\xi)y_1'(\xi) = \frac{1}{\alpha(\xi)}.$$

Together with $(*)$, we know that

$$A(\xi) = \frac{y_2(\xi)}{\alpha(\xi)W(\xi)}, \quad B(\xi) = \frac{y_1(\xi)}{\alpha(\xi)W(\xi)},$$

where W is the *Wronskian*

$$W = y_1 y_2' - y_2 y_1'.$$

Hence, we know that

$$G(x,\xi) = \frac{1}{\alpha(\xi)W(\xi)} \begin{cases} y_2(\xi)y_1(x) & a \le x \le \xi \\ y_1(\xi)y_2(x) & \xi < x \le b. \end{cases}$$

Using the step function Θ, we can write this as

$$G(x,\xi) = \frac{1}{\alpha(\xi)W(\xi)}[\Theta(\xi - x)y_2(\xi)y_1(x) + \Theta(x-\xi)y_1(\xi)y_2(x)].$$

So our general solution is

$$\begin{aligned} y(x) &= \int_a^b G(x,\xi)f(\xi)\,\mathrm{d}\xi \\ &= \int_x^b \frac{f(\xi)}{\alpha(\xi)W(\xi)} y_2(\xi)y_1(x)\,\mathrm{d}\xi + \int_a^x \frac{f(\xi)}{\alpha(\xi)W(\xi)} y_1(\xi)y_2(x)\,\mathrm{d}\xi. \end{aligned}$$

4.5.2.1 Use Green's Functions to solve Initial Value Problems

Consider $\mathcal{L}y = f(t)$ subject to $y(t = t_0) = 0$ and $y'(t = t_0) = 0$. We let y_a y_2 be a basis of solutions to $\mathcal{L}y = 0$. Then we have

$$G(t,\tau) = \begin{cases} A(\tau)y_1(t) + B(\tau)y_2(t) & t_0 \le t < \tau \\ C(\tau)y_1(t) + D(\tau)y_2(t) & t > \tau. \end{cases}$$

With initial conditions giving $A = b = 0$, and the continuity with jump conditions giving:

$$\begin{aligned} 0 &= C(\tau)t_1(t) + D(\tau)y_2(t) \\ \frac{1}{\alpha(\tau)} &= C(\tau)y_1'(\tau) + D(\tau)y_2'(\tau). \end{aligned}$$

Then we can solve to get:

$$y(t) = \int_{t_0}^{\infty} G(t,\tau)f(\tau)\,\mathrm{d}\tau = \int_{t_0}^{t} G(t,\tau)f(\tau)\,\mathrm{d}\tau,$$

4.5.3 Green's Functions for $\mathbb{R}^n$

4.5.3.1 Heat Equation

Suppose ϕ satisfies the inhomogeneous, forced, heat equation

$$\partial_t \phi - D\nabla^2 \phi = F(x,t),$$

but with homogeneous initial conditions $\phi|_{t=0} = 0$. Physically, this represents having an external heat source somewhere, starting at zero.

Note solving this solves heat equation completely. For inhomogeneous equation with inhomogeneous boundaries, we separate the differential equation into 2, and

solve the first one with inhomogeneous equation but homogeneous boundary and the second one with homogeneous equation but inhomogeneous boundary.

We take the fourier transform to reach that:

$$\partial_t \tilde{\phi}(k,t) + Dk^2 \tilde{\phi}(k,t) = \tilde{F}(k,t),$$

with the initial condition

$$\tilde{\phi}(k,0) = 0.$$

We solve this using an integrating factor to get:

$$\tilde{\phi}(k,t) = e^{-Dk^2 t} \int_0^t e^{Dk^2 u} \tilde{F}(k,u) \, \mathrm{d}u.$$

Now we look at this for Green's function, which we define as:

$$[\partial_t - D\nabla_x^2] G(x,t;y,\tau) = \delta(x-y)\delta(t-\tau).$$

So the Fourier transform with respect to x gives

$$\tilde{G}(k,t,y,\tau) = e^{-Dk^2 t} \int_0^t e^{Dk^2 u} e^{iky} \delta(t-\tau) \, \mathrm{d}u,$$

where e^{iky} is just the Fourier transform of $\delta(x-y)$. This is equal to

$$\tilde{G}(k,t;y,\tau) = \begin{cases} 0 & t < \tau \\ e^{-iky} e^{-Dk^2(t-\tau)} & t > \tau \end{cases} = \Theta(t-\tau) e^{-iky} e^{-Dk^2(t-\tau)}.$$

Reverting the Fourier transform, we get

$$G(x,t;y,\tau) = \frac{\Theta(t-\tau)}{2\pi} \int_{\mathbb{R}} e^{ik(x-y)} e^{-Dk^2(t-\tau)} \, \mathrm{d}x$$

This integral is just the inverse Fourier transform of the Gaussian with a phase shift. So we end up with

$$G(x,t;y,\tau) = \frac{\Theta(t-\tau)}{\sqrt{4\pi D(t-\tau)}} \exp\left(-\frac{(x-y)^2}{4D(t-\tau)}\right)$$

So the solution we seek is just:

$$\phi(x,t) = \int_0^t \int_{\mathbb{R}} F(y,\tau) G(x,t;y,\tau) \, \mathrm{d}y \, \mathrm{d}\tau.$$

4.5.3.2 Wave Equation

Suppose $\phi : \mathbb{R}^n \times [0,\infty) \to \mathbb{C}$ solves the inhomogeneous wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = F(\mathbf{x},t)$$

with

$$\phi(\mathbf{x},0) = \frac{\partial}{\partial t} \phi(\mathbf{x},0) = 0.$$

We look for a Green's function $G_n(\mathbf{x}, t; \mathbf{y}, \tau)$ that solves

$$\frac{\partial G_n}{\partial t} - c^2 \nabla^2 G_n = \delta(t-\tau)\delta^{(n)}(\mathbf{x}-\mathbf{y}) \qquad (*)$$

with the same initial conditions

$$G_n(\mathbf{x}, 0, \mathbf{y}, \tau) = \frac{\partial}{\partial t} G_n(\mathbf{x}, 0, \mathbf{y}, \tau) = 0.$$

Just as before, we take the Fourier transform of this equation with respect the spacial variables $\mathbf{x}$. We get

$$\frac{\partial}{\partial t}\tilde{G}_n + c^2|\mathbf{k}|^2 \tilde{G}_n = \delta(t-\tau)e^{-i\mathbf{k}\cdot\mathbf{y}}.$$

where $\tilde{G}_n = \tilde{G}_n(\mathbf{k}, t, \mathbf{y}, \tau)$.

This is just an ordinary differential equation from the point of view of t, and is of the same type of initial value problem that we studied earlier, and the solution is

$$\tilde{G}_n(\mathbf{k}, t, \mathbf{y}, \tau) = \Theta(t-\tau)e^{-i\mathbf{k}\cdot\mathbf{y}} \frac{\sin|\mathbf{k}|c(t-\tau)}{|\mathbf{k}|c}.$$

To recover the Green's function itself, we have to compute the inverse Fourier transform, and find

$$G_n(\mathbf{x}, t; \mathbf{y}, \tau) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot\mathbf{x}} \Theta(t-\tau) e^{-i\mathbf{k}\cdot\mathbf{y}} \frac{\sin|\mathbf{k}|c(t-\tau)}{|\mathbf{k}|c} \, \mathrm{d}^n k.$$

Unlike the case of the heat equation, the form of the answer we get here does depend on the number of spatial dimensions n. For definiteness, we look at the case where $n = 3$, since our world (probably) has three dimensions. Then our Green's function is

$$G(\mathbf{x}, t; \mathbf{y}, \tau) = \frac{\Theta(t-\tau)}{(2\pi)^3 c} \int_{\mathbb{R}^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{\sin|\mathbf{k}|c(t-\tau)}{|\mathbf{k}|} \, \mathrm{d}^3 k.$$

We use spherical polar coordinates with the z-axis in k-space aligned along the direction of $\mathbf{x}-\mathbf{y}$. Hence $\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}) = kr\cos\theta$, where $r = |\mathbf{x}-\mathbf{y}|$ and $k = |\mathbf{k}|$.

Note that nothing in our integral depends on φ, so we can pull out a factor of 2π, and get

$$G(\mathbf{x}, t; \mathbf{y}, \tau) = \frac{\Theta(t-\tau)}{(2\pi)^2 c} \int_0^\infty \int_0^\pi e^{ikr\cos\theta} \frac{\sin kc(t-\tau)}{k} k^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}k$$

The next integral to do is the θ integral, which is straightforward since it is an exact differential. So we get

$$\begin{aligned}
&= \frac{\Theta(t-\tau)}{(2\pi)^2 c} \int_0^\infty \left[\frac{e^{ikr} - e^{-ikr}}{ikr}\right] \frac{\sin kc(t-\tau)}{k} k^2 \, \mathrm{d}k \\
&= \frac{\Theta(t-\tau)}{(2\pi)^2 icr} \left[\int_0^\infty e^{ikr}\sin k\alpha \, \mathrm{d}k - \int_0^\infty e^{-ikr}\sin k\alpha \, \mathrm{d}k\right] \\
&= \frac{\Theta(t-\tau)}{2\pi icr} \left[\frac{1}{2\pi}\int_{-\infty}^\infty e^{ikr}\sin k\alpha \, \mathrm{d}k\right] \\
&= \frac{\Theta(t-\tau)}{2\pi icr} \mathcal{F}^{-1}[\sin k\alpha],
\end{aligned}$$

where we let $\alpha = c(t-\tau)$.

Now recall $\mathcal{F}[\delta(x-\alpha)] = e^{ik\alpha}$. So

$$\mathcal{F}^{-1}[\sin k\alpha] = \mathcal{F}^{-1}\left[\frac{e^{ik\alpha} - e^{-ik\alpha}}{2i}\right] = \frac{1}{2i}[\delta(x-\alpha) - \delta(x+\alpha)].$$

Hence our Green's function is

$$G(\mathbf{x}, t; \mathbf{y}, \tau) = -\frac{\Theta(t-\tau)}{4\pi c|\mathbf{x}-\mathbf{y}|}\Big[\delta(|\mathbf{x}-\mathbf{y}| - c(\mathbf{t}-\tau)) - \delta(|\mathbf{x}-\mathbf{y}| + c(t-\tau))\Big].$$

Now we look at our delta functions. The step function is non-zero only if $t > \tau$. Hence $|\mathbf{x}-\mathbf{y}| + c(t-\tau)$ is always positive. So $\delta(|\mathbf{x}-\mathbf{y}| + c(t-\tau))$ does not contribute. On the other hand, $\delta(|\mathbf{x}-\mathbf{y}| - c(t-\tau))$ is non-zero only if $t > \tau$. So $\Theta(t-\tau)$ is always positive in this region. So we can write our Green's function as

$$G(\mathbf{x}, t; \mathbf{y}, \tau) = -\frac{1}{4\pi c}\frac{1}{|\mathbf{x}-\mathbf{y}|}\delta(|\mathbf{x}-\mathbf{y}| - c(t-\tau)).$$

As always, given our Green's function, the general solution to the forced equation

$$\frac{\partial^2\phi}{\partial t^2} - c^2\nabla^2\phi = F(\mathbf{x}, t)$$

is

$$\phi(\mathbf{x}, t) = -\int_0^\infty \int_{\mathbb{R}^3} \frac{F(\mathbf{y}, \tau)}{4\pi c|\mathbf{x}-\mathbf{y}|}\delta(|\mathbf{x}-\mathbf{y}| - c(t-\tau))\ \mathrm{d}^3y.$$

We can use the delta function to do one of the integrals. It is up to us which integral we do, but we pick the time integral to do. Then we get

$$\phi(\mathbf{x}, t) = -\frac{1}{4\pi c^2}\int_{\mathbb{R}^3} \frac{F(\mathbf{y}, t_{\mathrm{ret}})}{|\mathbf{x}-\mathbf{y}|}\ \mathrm{d}^3y,$$

where

$$t_{\mathrm{ret}} = t - \frac{|\mathbf{x}-\mathbf{y}|}{c}.$$

This shows that the effect of the forcing term at some point $\mathbf{y} \in \mathbb{R}^3$ affects the solution ϕ at some other point $\mathbf{x}$ not instantaneously, but only after time $|\mathbf{x}-\mathbf{y}|/c$ has elapsed. This, again, tells us that information travels at speed c.

4.5.3.3 Poisson's equation

Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ satisfy the Poisson's equation

$$\nabla^2\phi = -F,$$

where $F(\mathbf{x})$ is a forcing term.

The fundamental solution to this equation is defined to be $G_3(\mathbf{x}, \mathbf{y})$, where

$$\nabla^2 G_3(\mathbf{x}, \mathbf{y}) = \delta^{(3)}(\mathbf{x}-\mathbf{y}).$$

By rotational symmetry, $G_3(\mathbf{x}, \mathbf{y}) = G_3(|\mathbf{x}-\mathbf{y}|)$. Integrating over a ball

$$B_r = \{|\mathbf{x}-\mathbf{y}| \le r, \mathbf{x} \in \mathbb{R}^3\},$$

we have

$$\begin{aligned} 1 &= \int_{B_r} \nabla^2 G_3 \, \mathrm{d}V \\ &= \int_{\partial B_r} \mathbf{n} \cdot \nabla G_3 \, \mathrm{d}S \\ &= \int_{S^2} \frac{\mathrm{d}G_3}{\mathrm{d}r} r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= 4\pi r^2 \frac{\mathrm{d}G_3}{\mathrm{d}r}. \end{aligned}$$

So we know

$$\frac{\mathrm{d}G_3}{\mathrm{d}r} = \frac{1}{4\pi r^2},$$

and hence

$$G_3(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} + c.$$

We often set $c = 0$ such that

$$\lim_{|\mathbf{x}| \to \infty} G_3 = 0.$$

4.5.4 Green's identities

To make use of this fundamental solution in solving Poisson's equation, we first obtain some useful identities.

Suppose $\phi, \psi : \mathbb{R}^3 \to \mathbb{R}$ are both smooth everywhere in some region $\Omega \subseteq \mathbb{R}^3$ with boundary $\partial\Omega$. Then

$$\int_{\partial\Omega} \phi \mathbf{n} \cdot \nabla\psi \, \mathrm{d}S = \int_\Omega \nabla \cdot (\phi \nabla\psi) \, \mathrm{d}V = \int_\Omega \phi\nabla^2\psi + (\nabla\phi) \cdot (\nabla\psi) \, \mathrm{d}V.$$

So we get

Proposition (Green's first identity)**.**

$$\int_{\partial\Omega} \phi \mathbf{n} \cdot \nabla\psi \, \mathrm{d}S = \int_\Omega \phi\nabla^2\psi + (\nabla\phi) \cdot (\nabla\psi) \, \mathrm{d}V.$$

Of course, this is just an easy consequence of the divergence theorem, but when Green first came up with this, divergence theorem hasn't existed yet.

Similarly, we obtain

$$\int_{\partial\Omega} \psi \mathbf{n} \cdot \nabla\phi \, \mathrm{d}S = \int_\Omega \psi\nabla^2\phi + (\nabla\phi) \cdot (\nabla\psi) \, \mathrm{d}V.$$

Subtracting these two equations gives

Proposition (Green's second identity)**.**

$$\int_\Omega \phi\nabla^2\psi - \psi\nabla^2\phi \, \mathrm{d}V = \int_{\partial\Omega} \phi \mathbf{n} \cdot \nabla\psi - \psi \mathbf{n} \cdot \nabla\phi \, \mathrm{d}S.$$

Note. The second identity is often used to prove uniqueness of solution to a Poisoon or Laplace's equation.

Using the Green's function

We wish to apply this result to the case $\psi = G_3(|\mathbf{x} - \mathbf{y}|)$. However, recall that when deriving Green's identity, we assumed ϕ and ψ are smooth everywhere in our domain. However, our Green's function is singular at $\mathbf{x} = \mathbf{y}$, but we want to integrate over this region as well. So we need to do this carefully. Because of the singularity, we want to take

$$\Omega = B_r - B_\varepsilon = \{\mathbf{x} \in \mathbb{R}^3 : \varepsilon \leq |\mathbf{x} - \mathbf{y}| \leq R\}.$$

In other words, we remove a small region of radius ε centered on $\mathbf{y}$ from the domain.

In this choice of Ω, it is completely safe to use Green's identity, since our Green's function is certainly regular everywhere in this Ω. First note that since $\nabla^2 G_3 = 0$ everywhere except at $\mathbf{x} = \mathbf{y}$, we get

$$\int_\Omega \phi \nabla^2 G_3 - G_3 \nabla^2 \phi \, \mathrm{d}V = -\int_\Omega G_3 \nabla^2 \phi \, \mathrm{d}V$$

Then Green's second identity gives

$$\begin{aligned} -\int_\Omega G_3 \nabla^2 \phi \, \mathrm{d}V &= \int_{S_r^2} \phi(\mathbf{n} \cdot \nabla G_3) - G_3(\mathbf{n} \cdot \phi) \, \mathrm{d}S \\ &\quad + \int_{S_\varepsilon^2} \phi(\mathbf{n} \cdot \nabla G_3) - G_3(\mathbf{n} \cdot \phi) \, \mathrm{d}S \end{aligned}$$

Note that on the inner boundary, we have $\mathbf{n} = -\hat{\mathbf{r}}$. Also, at S_ε^2, we have

$$G_3|_{S_\varepsilon^2} = -\frac{1}{4\pi\varepsilon}, \quad \left.\frac{\mathrm{d}G_3}{\mathrm{d}r}\right|_{S_\varepsilon^2} = \frac{1}{4\pi\varepsilon^2}.$$

So the inner boundary terms are

$$\begin{aligned} &\int_{S_\varepsilon^2} \Big(\phi(\mathbf{n} \cdot \nabla G_3) - G_3(\mathbf{n} \cdot \phi)\Big)\varepsilon^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= -\frac{\varepsilon^2}{4\pi\varepsilon^2} \int_{S_\varepsilon^2} \phi \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi + \frac{\varepsilon^2}{4\pi\varepsilon} \int_{S_\varepsilon^2} (\mathbf{n} \cdot \nabla\phi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \end{aligned}$$

Now the final integral is bounded by the assumption that ϕ is everywhere smooth. So as we take the limit $\varepsilon \to 0$, the final term vanishes. In the first term, the ε's cancel. So we are left with

$$\begin{aligned} &= -\frac{1}{4\pi} \int_{S_\varepsilon^2} \phi \, \mathrm{d}\Omega \\ &= -\bar{\phi} \\ &\to -\phi(\mathbf{y}) \end{aligned}$$

where $\bar{\phi}$ is the average value of ϕ on the sphere.

Now suppose $\nabla^2 \phi = -F$. Then this gives

Proposition (Green's third identity)**.**

$$\phi(\mathbf{y}) = \int_{\partial\Omega} \phi(\mathbf{n} \cdot \nabla G_3) - G_3(\mathbf{n} \cdot \nabla\phi) \, \mathrm{d}S - \int_\Omega G_3(\mathbf{x}, \mathbf{y}) F(\mathbf{x}) \, \mathrm{d}^3 x.$$

This expresses ϕ at any point $\mathbf{y}$ in Ω in terms of the fundamental solution G_3, the forcing term F and boundary data. In particular, if the boundary values of ϕ and $\mathbf{n} \cdot \nabla \phi$ vanish as we take $r \to \infty$, then we have

$$\phi(\mathbf{y}) = -\int_{\mathbb{R}^3} G_3(\mathbf{x}, \mathbf{y}) F(\mathbf{x}) \, \mathrm{d}^3 x.$$

So the fundamental solution *is* the Green's function for Poisson's equation on $\mathbb{R}^3$.

4.5.4.1 Method of Images

Example. Suppose a chimney produces smoke such that the density $\phi(\mathbf{x}, t)$ of smoke obeys

$$\partial_t \phi - D\nabla^2 \phi = F(\mathbf{x}, t).$$

The left side is just the heat equation, modelling the diffusion of smoke, while the right forcing term describes the production of smoke by the chimney.

If this were a problem for $\mathbf{x} \in \mathbb{R}^3$, then the solution is

$$\phi(\mathbf{x}, t) = \int_0^t \int_{\mathbb{R}^3} F(\mathbf{y}, \tau) S_3(\mathbf{x} - \mathbf{y}, t - \tau) \, \mathrm{d}^3 \mathbf{y} \, \mathrm{d}\tau,$$

where

$$S_3(\mathbf{x} - \mathbf{y}, t - \tau) = \frac{1}{[4\pi D(t - \tau)]^{3/2}} \exp\left(-\frac{|\mathbf{x} - \mathbf{y}|^2}{4D(t - \tau)}\right).$$

This is true only if the smoke can diffuse in all of $\mathbb{R}^3$. However, this is not true for our current circumstances, since smoke does not diffuse into the ground.

To account for this, we should find a Green's function that obeys

$$\mathbf{n} \cdot \nabla G|_{z=0} = 0.$$

This says that there is no smoke diffusing in to the ground.

This is achieved by picking

$$G(\mathbf{x}, t; \mathbf{y}, \tau) = \Theta(t - \tau)[S_3(\mathbf{x} - \mathbf{y}, t - \tau) + S_3(\mathbf{x} - \mathbf{y}^R, t - \tau)].$$

We can directly check that this obeys

$$\partial_t D^2 \nabla^2 G = \delta(t - \tau)\delta^3(\mathbf{x} - \mathbf{y})$$

when $\mathbf{x} \in \Omega$, and also

$$\mathbf{n} \cdot \nabla G | z_0 = 0.$$

Hence the smoke density is given by

$$\phi(\mathbf{x}, t) = \int_0^t \int_\Omega F(\mathbf{y}, \tau)[S_3(\mathbf{x} - \mathbf{y}, t - \tau) + S_3(\mathbf{x} - \mathbf{y}^R, t - \tau)] \, \mathrm{d}^3 y.$$

We can think of the second term as the contribution from a "mirror chimney". Without a mirror chimney, we will have smoke flowing into the ground. With a mirror chimney, we will have equal amounts of mirror smoke flowing up from the ground, so there is no net flow. Of course, there are no mirror chimneys in reality. These are just artifacts we use to find the solution we want.

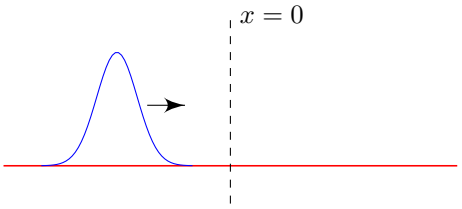
Example. Suppose we want to solve the wave equation in the region (x, t) such that $x > 0$ with boundary conditions

$$\phi(x, 0) = b(x), \quad \partial_t(x, 0) = 0, \quad \partial_x \phi(0, t) = 0.$$

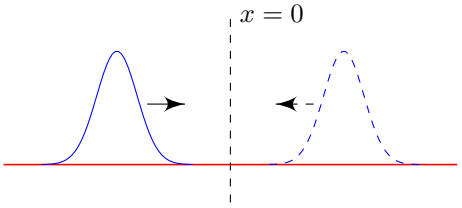
On $\mathbb{R}^{1,1}$ d'Alembert's solution gives

$$\phi(x, t) = \frac{1}{2}[b(x - ct) + b(x + ct)]$$

This is not what we want, since eventually we will have a wave moving past the $x = 0$ line.



To compensate for this, we introduce a mirror wave moving in the opposite direction, such that when as they pass through each other at $x = 0$, there is no net flow across the boundary.



More precisely, we include a *mirror initial condition* $\phi(x, 0) = b(x) + b(-x)$, where we set $b(x) = 0$ when $x < 0$. In the region $x > 0$ we are interested in, only the $b(x)$ term will contribute. In the $x < 0$ region, only $x > 0$ will contribute. Then the general solution is

$$\phi(x, t) = \frac{1}{2}[b(x + ct) + b(x + c) + b(-x - ct) + b(-x + ct)].$$

4.6 Well-posedness and Method of Characteristics

4.6.1 Well-Posedness

Definition (Well-posed problem)**.** A partial differential equation problem is said to be well-posed if its Cauchy data means

(i) A solution exists;

(ii) The solution is unique;

(iii) A "small change" in the Cauchy data leads to a "small change" in the solution.

Its *Cauchy Data* is the set of initial and boundary conditions that the differential equation needs to satisfy.

This is important. If we look at the reverse heat equation, any very small change in the initial data curve would be amplified exponentially (opposite of the actual hear equation), and this is really not what we want.

4.6.2 Method of Characteristics

Note. This section, to a lot of people (including me), is particularly confusing. You can just get used to it by doing more problems, but it is nicer if you understand the concepts behind it. I will remove some rigidity in definitions in exchange for clarity

Now we start by parametrizing a curve $(x(s), y(s))$. Clearly, its tangent vector is $(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s})$.

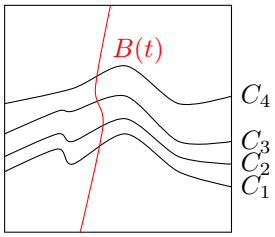
Consider a vector field. Imagine a vector field as a vector associating to each point. The only use of the vector field we will ever encounter in life is the use for flow. So the vector associated at that point tells you exactly how stuff is flowing (magnitude and direction).

Therefore, we would like to look at the curves of motion of a particle being put into this vector field flow. Now since the velocity vector of a curve is just its tangent vector, we define the following class of curves:

Definition (Integral curve)**.** Let $\mathbf{V}(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field. The *integral curves* associated to $\mathbf{V}$ are curves whose tangent $\left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}\right)$ is just $\mathbf{V}(x, y)$.

This means that the curves describe some particle in motion in the field, as at every point, they correspond to the vector field flow.

For sufficiently regular vector fields, we can fill the space with different curves. We will parametrize which curve we are on by the parameter t. More precisely, we have a curve $B = (x(t), y(t))$ that is *transverse* (ie. nowhere parallel) to our family of curves, and we can label the members of our family by the value of t at which they intersect B.



We can thus label our family of curves by $(x(s, t), y(s, t))$. If the Jacobian

$$J = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \neq 0,$$

then we can invert this to find (s, t) as a function of (x, y), ie. at any point, we know which curve we are on, and how far along the curve we are.

Now (s, t) is thus a coordinate system and it turns out, for specific differential equations, this makes life easier.

4.6.2.1 Method of Characteristics in 1D

Note (Fluid Takers)**.** In this section, I am going to use some fluid dynamics notation. This is because there is a direct analogy from this to streamfunctions, with ϕ as the streamfunction (and as this is time-independent, the contours of the streamfunction are the particle curves), and the vector field $\mathbf{u}$ as the fluid flow.

Now we consider

$$a(x,y)\frac{\partial \phi}{\partial x} + b(x,y)\frac{\partial \phi}{\partial y} = 0.$$

$$\mathbf{u}(x,y) = \begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix}.$$

Then we can write the differential equation as

$$\mathbf{u} \cdot \nabla \phi = 0.$$

Along any particular integral curve of $\mathbf{V}$, we have

$$\frac{\partial \phi}{\partial s} = \frac{\mathrm{d}x(s)}{\mathrm{d}s}\frac{\partial \phi}{\partial x} + \frac{\mathrm{d}y(s)}{\mathrm{d}s}\frac{\partial \phi}{\partial y} = \mathbf{u} \cdot \nabla \phi,$$

where the integral curves of $\mathbf{u}$ are determined by

$$\left.\frac{\partial x}{\partial s}\right|_t = a(x,y), \quad \left.\frac{\partial y}{\partial s}\right|_t = b(x,y).$$

This is known as the characteristic equation.

Hence our partial differential equation just becomes the equation

$$\left.\frac{\partial \phi}{\partial s}\right|_t = 0.$$

To get ourselves a well-posed problem, we have to specify our boundary data along a transverse curve B. We pick our transverse curve as $s = 0$, and we suppose we are given the initial data

$$\phi(0,t) = h(t).$$

Since ϕ does not vary with s, our solution automatically is

$$\phi(s,t) = h(t).$$

Then $\phi(x,y)$ is given by inverting the characteristic equations to find $t(x,y)$.

Now to clear up the topic even further, let's look at an example:

Example. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ solve the inhomogeneous partial differential equation

$$\partial_x \phi + 2\partial_y \phi = ye^x$$

with $\phi(x,x) = \sin x$. It's ok. Even if it is inhomogeneous, we can still use the method of characteristics. We have

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

So the characteristic curves obey

$$\frac{\mathrm{d}x}{\mathrm{d}s} = 1, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = 2.$$

This gives

$$x = s + c, \quad y = 2s + d$$

Ok. Now we have some freedom in picking the transverse curve B, so the next step is not deterministic (there are many curves that would work). But why work with something hard? We would pick the curve $s = 0$. This can be seen "intuitively". We are basically picking the y-axis in (s, t) coordinates, which is *kind of* guaranteed to be transverse to the solutions, because a solution parallel to that would have infinite slope.

But we still have a problem. What if B intersects every particle curve more than once? Then the value of the particle curve on the multiple intersections cannot be specified freely. It must correspond to the differential equation, because clearly otherwise the particle curve doesn't satisfy the differential equation.

But here it should have no problem. We want the curve $s = 0$ to parametrize the boundary condition, so we need $x = y = t$ (or more generally multiples of t, but why consider more complicated functions than needed?)

So $c = d = t$.We can invert this to obtain the relations

$$s = y - x, \quad t = 2x - y.$$

The partial differential equation now becomes

$$\left.\frac{\mathrm{d}\phi}{\mathrm{d}s}\right|_t = \mathbf{u} \cdot \nabla\phi = ye^x = (2s + t)e^{s+t}$$

Note that this is just an ordinary differential equation in s because t is held constant. We have the boundary conditions

$$\phi(s = 0, t) = \sin t.$$

So we get

$$\phi(x(s, t), y(s, t)) = (2 - t)e^t(1 - e^s) + \sin t + 2se^{s+t}.$$

Putting it in terms of x and y, we have

$$\phi(x, y) = (2 - 2x + y)e^{2x-y} + \sin(2x - y) + (y - 2)e^x.$$

4.6.2.2 Characteristics in 2D

Now we move away from intuition and into higher orders. In this, we have:

Definition (Symbol and principal part)**.** Let $\mathcal{L}$ be the general 2nd order differential operator on $\mathbb{R}^n$. We can write it as

$$\mathcal{L} = \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i(x)\frac{\partial}{\partial x^i} + c(X),$$

where $a^{ij}(x), b^i(x), c(x) \in \mathbb{R}$ and $a^{ij} = a^{ji}$ (wlog).

We define the *symbol* $\sigma(\mathbf{k}, x)$ of $\mathcal{L}$ to be

$$\sigma(\mathbf{k}, x) = \sum_{i,j=1}^{n} a^{ij}(x)k_i k_j + \sum_{i=1}^{n} b^i(x)k_i + c(x).$$

So we just replace the derivatives by the variable k.

The *principal part* of the symbol is the leading term

$$\sigma^p(\mathbf{k}, x) = \sum_{i,j=1}^{n} a^{ij}(x)k_i k_j.$$

In general, for each x, we can write

$$\sigma^p(\mathbf{k}, x) = \mathbf{k}^T A \mathbf{k},$$

where $A(x)$ has elements $a^{ij}(x)$. Recall that a real symmetric matrix (such as A) has all real eigenvalues. So we define the following:

Definition (Elliptic, hyperbolic, ultra-hyperbolic and parabolic differential operators)**.** We say a differential operator $\mathcal{L}$ is *elliptic* if all eigenvalues have the same sign.

We say it is *hyperbolic* if all but one eigenvalues have the same sign.

We say it is *parabolic* if A is degenerate, ie. has a zero eigenvalue.

4.6.2.3 Characteristic Surface

Given a differential operator $\mathcal{L}$, let

$$f(x^1, x^2, \cdots, x^n) = 0$$

define a surface $C \subseteq \mathbb{R}^n$. We say C is *characteristic* if

$$\sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} = (\nabla f)^T A (\nabla f) = 0.$$

In the case where we only have two dimensions, a characteristic surface is just a curve. Now we let:

$$\mathcal{L} = a(x,y)\frac{\partial^2}{\partial x^2} + 2b(x,y)\frac{\partial^2}{\partial x \partial y} + c(x,y)\frac{\partial^2}{\partial y^2} + d(x,y)\frac{\partial}{\partial x} + e(x,y)\frac{\partial}{\partial y} + f(x,y).$$

As f is constant on each characteristic, we have:

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Hence, we can compute

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{-b \pm \sqrt{b^2 - ac}}{a}.$$

We now see explicitly how the type of the differential equation influences the number of characteristics — if $b^2 - ac > 0$, then we obtain two distinct differential equations and obtain two solutions; if $b^2 - ac = 0$, then we only have one equation; if $b^2 - ac < 0$, then there are no real characteristics.

Example. Consider the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = 0$$

on $\mathbb{R}^{1,1}$. Then the equation is hyperbolic everywhere, and the characteristic curves are $x \pm ct = \text{const}$. Let's look for a solution to the wave equation that obeys

$$\phi(x, 0) = f(x), \quad \partial_t \phi(x, 0) = g(x).$$

Now put $u = x - ct, v = x + ct$. Then the wave equation becomes

$$\frac{\partial^2}{\partial u \partial v} = 0.$$

So the general solution to this is

$$\phi(x,t) = G(u) + H(v) = G(x-ct) + H(x+ct).$$

The initial conditions now fix these functions

$$f(x) = G(x) + H(x), \quad g(x) = -cG'(x) + cH'(x).$$

Solving these, we find

$$\phi(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y.$$

This is d'Alembert's solution to the $1+1$ dimensional wave equation.

Appendices

A Linear Algebra Review

Here are some linear algebra terms we used (without really defining) in the notes.

Definition (Vector space)**.** A *vector space* over $\mathbb{C}$ (or $\mathbb{R}$) is a set V with an operation $+$ which obeys

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity)

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associativity)

(iii) There is some $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u}$ (identity)

We can also multiply vectors by a scalars $\lambda \in \mathbb{C}$, which satisfies

(i) $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$ (associativity)

(ii) $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$ (distributivity in V)

(iii) $(\lambda + \mu)\mathbf{u} = \lambda\mathbf{u} + \lambda\mathbf{v}$ (distributivity in $\mathbb{C}$)

(iv) $1\mathbf{v} = \mathbf{v}$ (identity)

Definition (Inner product)**.** An *inner product* on V is a map $(\cdot, \cdot) : V \times V \to \mathbb{C}$ that satisfies

(i) $(\mathbf{u}, \lambda\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v})$ (linearity in second argument)

(ii) $(\mathbf{u}, \mathbf{v} + \mathbf{w}) = (\mathbf{u}, \mathbf{v}) + (\mathbf{u}, \mathbf{w})$ (additivity)

(iii) $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})^*$ (conjugate symmetry)

(iv) $(\mathbf{u}, \mathbf{u}) \geq 0$, with equality iff $\mathbf{u} = \mathbf{0}$ (positivity)

Note that the positivity condition makes sense since conjugate symmetry entails that $(\mathbf{u}, \mathbf{u}) \in \mathbb{R}$.

The inner product in turn defines a norm $\|\mathbf{u}\| = \sqrt{(\mathbf{u}, \mathbf{u})}$ that provides the notion of length and distance.

Definition (Basis)**.** A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ form a *basis* of V iff any $\mathbf{u} \in V$ can be uniquely written as a linear combination

$$\mathbf{u} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i$$

for some scalars λ_i. The *dimension* of a vector space is the number of basis vectors in its basis.

A basis is *orthogonal* (with respect to the inner product) if $(\mathbf{v}_i, \mathbf{v}_j) = 0$ whenever $i \neq j$. it is *orthonormal* if additionally all the vectors have a norm of 1.