Graph Theory

Michael Li

March 9, 2017

For Later

Contents

1 Planar Graphs

Given a plane graph G, $\mathbb{R}^2 - G$ splits up into connected regions called *faces*. THe *boundary* of a face consists of the vertices and edges that touch it.

Note. - The boundary of a face need not be a cycle!

- THe boundary of a face need not be connected! Consider two concentric squares as an example.
- Both sides of an edge can be on the same face.

1.1 Formal Definitions

For $x, y \in \mathbb{R}^2$, $x \neq y$, A *polygonal arc* from x to y is a finite union of closed straight-line segments $\overline{x_1\ x_2} \cup \cdots \cup \overline{x_{k-1}\ x_k}$ with $x_1 = x$, $x_k = y$, disjoint except for $\overline{x_i\ x_{i+1}} \cap \overline{x_{i+1}\ x_{i+2}} = \{x_{i+1}\}$.

For Graph G, vertices $v_1, \cdots, v_n$, a *drawing* of G consists of distinct points $x_1, \cdots x_n \in \mathbb{R}^2$, and for each $v_i, v_j \in E$, a polygonal arc $p_{i,j}$ from $x_i \to x_j$, such that $p_{i,j} \cap p_{k,l} = \emptyset$ for i, j, k, l distinct and $p_{i,j} \cap p_{i,k} = (x_i)$.

For $x, y \in \mathbb{R}^2 - G$, write $x \sim y$ if there exists a polygonal arc in $\mathbb{R}^2 - G$ from x to y. The components of $\sim$ are the *faces* of G. The *boundary* of a face F is G intersect the closure of F.

Theorem (Euler's Formula)**.** Let G be a connected plane graph with n vertices, m edges, and f faces. Then $n - m + f = 2$.

Proof. If G is a tree, then $m = n - 1$, and $f = 1$, so $n - m + f = 2$.

If G is not a tree, then we choose an edge l on a cycle C. Then $G - l$ connected. With n vertices, $m - 1$ edges, $f - 1$ faces, so by induction, we can reduce it to the tree case, and thus we are done. □

Theorem. Let G be a planar graph. Then $m \leq 3n - 6$.

Proof. If we sum for each face in G. THe number of edges in its boundary, we obtain $\geq 3f$, because each face has at least 3 edges. Also each edge is counted at most 2 times, so we have $\leq 2m$. So $3f \leq 2m$. Now WLOG G connected, so $n - m + f = 2$. Then $n - m + \frac{2}{3}m \geq 2$, or $m \leq 3n - 6$. □

Proposition. $K_{3,3}$ is not planar.

Remark. This is equivalent to connecting 3 houses to 3 utility companies without pipes crossing.

Proof. If drawn in plane, each face has ≥ 4 edges, as $K_{3,3}$ is bipartite. So $4f \leq 2m$ as in the proof of Euler's formula. Thus $n - m + \frac{m}{2} \geq 2$, so $m \leq 2n - 4$, which is false for $K_{3,3}$. □

From this, we can note that if we can eliminate 3-cycles, 4-cycles,$\cdots$, then we can get tighter bounds than $3n - 6$.

Theorem (Kuratowski's Theorem)**.** G is planar iff G does not contain a subdivided K_5 or $K_{3,3}$.

This is not proven in the course, but this provides a good summary for planar graphs.

2 Connectivity and Matchings

Let G be a bipartite graph, vertex classes X and Y. A *matching* from X to Y is a family of edges so that $X \to X' \subset Y$ is injective.

When do you have a matching? Well if $d(x) = 0$ for some $x \in X$, then there is no matching. If there exists distinct $x_1, x_2 \in X$ with $P(x_1) = P(x_2) = \{y\}$, then no matching.

Similarly, if for some $A \subset X$ we have $|P(A)| < |A|$, where $P(x)$ is the set of elements in Y such that x is connected to, we have no matching.

Theorem (Hall's Marriage Theorem)**.** If G is bipartite, and vertex classes X, Y. Then G has a matching from X to $Y \Leftrightarrow |P(a)| \geq |A|$ for all $A \subset X$.

Proof. One direction is trivial as we proved it above. Now we prove the $\Leftarrow$ direction.

We use induction. Induct on $|X|$. For $|X| = 1$ this is clearly true. Now we ask this question:

$$\text{Do we have } |P(A)| > |A| \text{ for all } A \subset X, A \neq 0, X?$$

If yes, then we choose $x \in X$, $y \in P(x)$, and let $G' = G - x - y$. Then we clearly see G' has a matching by induction because for every $A \in X \backslash \{x\}$, $P'_G(A) \geq |A|$ as originally we had a strict inequality. Thus we are done by induction hypothesis.

If not, we seem to have a big problem. Let $G' = G[A \cup P(A)]$, $G'' = G[(X \backslash A) \cup (Y \backslash P(A))]$. We clearly see that G' has a matching from A to $P(A)$ by induction hypothesis and definition. But what about G''?

If we consider $B \in X \backslash A$, we add that to A. Then we see that $|P(B \cup A)| \geq |B| + |A|$, but $P(A) \in G'$ and thus by construction $P(B) \geq |B|$ and $P(B) \subset G''$. □

Now we have a second proof, using the min-cut max-flow theorem from IB Optimization:

Proof. We form a directed network by adding a source S, joined to each $x \in X$ by an edge of capacity 1, and a sink t joined to each $Y \in Y$ by an edge of capacity 1, and direct each $xy \in E(G)$ from X to Y with capacity ∞. Then an integer-valued flow of size $|X|$ is precisely a matching from X to Y. So by integrality theorem of max-flow min-cut, it is enough to check that each cut has capacity $\geq |X|$.

Ley $\{s\} \cup A \cup B$, where $A \subset X$, $B \subset Y$ be a cut. Wlog $\Gamma(A) \subset B$, otherwise we have a cut along a line with capacity ∞. So capacity of cut is $|X| - |A| + |B| \geq |X|$ by the condition. So we are done. □

Definition. A *matching of deficiency d* is a set of $|X| - d$ independent edges.

Corollary (Defect Hall Theorem)**.** Let G be bipartite, vertex classes X, Y. Then G has a matching from X to Y of deficiency $d \Leftrightarrow |\Gamma(A)| \geq |A| - d \forall A \subset X$.

Proof. Note that $\Rightarrow$ is trivial. Now we add new vertices $y_a, \cdots, y_d$ to Y, each connected to all of X. Then the graph formed has a matching from X to $Y \cup \{y_1, \cdots, y_v\}$ by Hall'stheorem, which yields a set of $\geq |X| - d$ independent edges in G itself. $\square$

2.1 Transversal

Definition. A transversal of sets $S_1, \cdots, S_n$ consists of distinct $x_1, \cdots, x_n$ with $x_i \in S_i$ for all i.

Corollary. Sets $S_1, \cdots S_n$ have a transverse if and only if $|\bigcup_{i \in A} S_i| \geq |A|$ for all $A \subset \{1, \cdots, n\}$.

Proof. Again, one direction is trivial. Now create the graph with $1, \cdots, n$ on one side and all possible elements on the other side, wlog $S_1, \cdots, S_n$ finite. Then we connect the i and x when x is an element of S_i. Then clearly the condition above translates to Hall's condition, and we are done. $\square$

Example. Let G be a finite group and H a subgroup. We have left cosets $g_1 H, \cdots g_n H$, and right cosets $Hg'_1, \cdots, Hg'_k$. Can we choose $g_1 \cdots g_k$ such that $g_1 H, \cdots g_k H$ are the left cosets and $Hg_1, \cdot Hg_k$ are the right cosets?

In other words, if $L_1, \cdots L_k$ are the left cosets and $R_1, \cdots R_k$ the right ones, is there a reordering such that $L_i \cap R_i \neq \emptyset$ for all i.

We now form bipartite G, vertex classes $\{1, \cdots, k\}$, $y = \{1, \cdots, k\}$ with $n \in X$ joined to $j \in Y$ if L_i meets R_j. Thus we seek a matching, where we connect all the edges ij such that $L_i \cap R_j \neq \emptyset$. Just by counting the numebr of elements, we see n left cosets must need n right cosets to cover them, so hall's condition is satisfied. We are done.

3 Connectivity

The idea we are exploring is how "connected" a graph is.

Definition. A connected graph G, with $|G| > 1$, has *connectivity* $\kappa(G)$ the smallest $|S|$ such that $S \subset V$, $G - S$ disconnected (or a single point). We say G is k-connected if $\kappa(G) \geq k$.

Example.
- No tree is 2-connected.
- A cycle is 2-connected.
- A cube is 3-connected.
- K_n is $(n-1)$-connected.

Note.
- We have $K(G) \leq \delta(G)$, where $\delta(G) = \min_{x \in G} d(x)$.

Proposition. content...

For distinct a, b in G, say $S \in V \backslash \{a, b\}$ separates a from b, or is an $a - b$ separator. If a, b in different components of $G - S$.

Theorem (Menger's Theorem)**.** Let a, b be distinct non-adjacent vertices of graph G. Suppose every $a - b$ separator has size $\geq k$, then $\exists k$ independent $a - b$ paths.

Remark.

- The converse statement is trivial.
- Equivelently, the minimum size of a separator is the maximum size of a set of independent paths.
- Non-adjacent is needed for definition of separator to work.
- Menger implies hall theorem. Given bipartite G, add vertices a and b thats connected to all of X and Y. Then a matching from X to Y is precisely a set of $|X|$ independent $a - b$ paths . So by menger, we just need to check that every separator has size $\geq |X|$. Let $A \subset X$ and $B \subset Y$ be an $a - b$ separator. Then $P(X \backslash A) \subset B$, so $|B| \geq |X \backslash A|$ by hall's condition, so $|A| + |B| \geq |X|$.

Proof. WLOG $k \geq 2$. We can assume that we need to prove that k is the minimal size of an $a - b$ separator. If it is false, let g be a minimal counterexample, being the one with minimal k and then minimal number of edges for given k.

Let S be an $a - b$ separator with $|S| = k$. Suppose that $S \not\subset \Gamma(a)$ and $S \not\subset \Gamma(b)$. Form G' from G by replacing the component of a in $G - S$ with one point a', joined to all of S, thus $e(G') < e(G)$, as $S \notin \Gamma(a)$.

Then G' has no $a' - b$ separator of size $< k$, so G' has a family of k $a' - b$ independent paths by induction, so in other words, G has paths $B_1, \cdots, B_k$ from b to S except at b.

By symmetry, we can do this starting from a, and connecting these, as S is a separator, we are done. ☐

Corollary. Let G be connected, $|G| > 1$, then G is k connected iff for all distinct $a, b \in G$, there exists k independent $a - b$ paths.

Proof. The $\Leftarrow$ direction is trivial. For $\Rightarrow$, if $ab \in E$, then let $G' = G - ab$, then G' is $k - 1$ connected so $G = G' + ab$ is k connected. If $ab \notin E$, then we are done by theorem above. ☐

Definition. For G connected, $|G| > 1$, the *edge-connectivity* of G, written $\lambda(G)$, is the smallest $|w|$ where $w \in E$ and $G - w$ are disconnected. we say it is *k-edge-connected* if $\lambda(G) \geq k$.

Theorem. Let G be connected, and $a, b \in G$. Then the minimum size of $w \in E$ that separates a from b is the maximum size of a family of edge disjoint a-b paths.

Proof. For a graph g, the line graph $L(G)$ has vertex set $E(G)$ and two vertexes are connected if the two edges meet in G. now we add a' and b' so that a' are connected to the edges (now vertices) of the graph G' that a was connected to and similarly for b'. Then the edge-disjoint paths in G corresponds to independent $a - b$ paths in G', and edge separators in G correspond to separators in G'. Then we are done by Menger's Theorem ☐

Corollary. Let $|G| > 1$ be connected. Then G is k connected iff for all distinct $a, b \in G$ there exist a family of k edge-disjoint $a - b$ paths.

4 Extremal Problems

Definition. An *Euler circuit* in G is a circuit (a walk $x_1, \cdots, x_k$ with $x_k = x_1$) that includes every edge exactly once. We say G is Eulerian if it has an euler circuit.

Theorem. Let G be a connected graph. Then G Eulerian $\Leftrightarrow$ all degrees of G are even.

Proof. $\Rightarrow$ If euler circuit passes through vertex x k times, then $d(x) = 2k$.

$\Leftarrow$ We use induction on $l(G)$: For $l(G) = 0$, it is trivial. Given connected G, $l(G) > 0$, and all degrees are even: Suppose G is not eulerian, and let C be a longest circuit without a repeated edge - note ($l(C) > 0$, as g is not a tree as all degrees ≥ 2, so G has a cycle). Let H be a component of $G - E(C)$ with $l(H) > 0$. Now $\forall x \in V(H)$ we have $d_H(x)$ so by induction step H has an euler circuit. But $V(H)$ meets $V(C)$ as G is connected. So C, C' shares a vertex, so we can combine C, C' to obtain a longer circuit than C. A contradiction. So we are done.

□

Definition. A *Hamilton cycle* in G is a cycle passing through each vertex. We say G is hamiltonian if it has a hamilton cyccle.

Theorem. Let G be a graph on n vertices ($n \geq 3$) with $\delta(G) \geq \frac{n}{2}$. Then G is hamiltonian.

Proof. Most have G connected. Indeed, if x, y non-adjacent then $P(x), P(y) < V - \{x, y\}$, whence $P(x) \cap P(y) \neq 0$.

Ley $x_1, \cdots x_l$ be longest path in G, and we have $l \geq 3$ as G is connected, $|G| \geq 3$. Wlog that G has no l-cycle. If $l = n$ then we are done. If $L < n$, then there exists $xy \in E$ with $x \in$ cycle and $y \not\in n$ cycle, so we obtain path on $l + 1$ vertices.

Thus $x_1 x_l \notin E$. Also for each $2 \leq i \leq l$,we cannot have both $x_1 x_i$ and $x_{i-1} x_l$ in E, as else we have l-cycle. Now $P(x_1), P(x_l) \in \{x_2, \cdots x_{l-1}\}$ by maximality. Now by definition $\delta(G) \geq \frac{n}{2}$, and $l \leq n$, so if x_1 is connected to $\frac{n}{2}$ of $x_2, \cdots x_{l-1}$ of these, then those shifted by one cannot be connected by x_l. But then x_l can only be connected to $l - \frac{n}{2} < \frac{n}{2}$ a contradiction. So we are done. □

Proposition. Let G be a connected graph of order n, $n \geq 3$, and let $k < n$. Then $\delta(G) \geq \frac{k}{2} \Rightarrow G \supset P_k$.

Proof. CHoose a longest path $x_1, \cdots x_l$, and suppose $l \leq k$. Then G has no l cycle by theorem above, and by the same reasoning as the theorem above, we have a contradiction. □

Theorem. Let G be a graph on n vertives with $l(G) > \frac{n(k-1)}{2}$. Then G has a P_k.

Proof. Given $G \not\supset P_k$, we want $l(G) \leq \frac{n(k-1)}{2}$. Without loss of generality $k < n$. We induct on n. Given G, $G \not\supset P_k$: Given G, $G \not\supset P_k$: Without loss of generality, assume G is connected, as if disconnected, we are done by induction on each disconnected component. Now choose $x \in G$ with $d(x) \leq \frac{k-1}{2}$ (exists by proposition above), but $G - x \not\supset P_k$, so $l(G - x) \leq \frac{(n-1)(k-1)}{2}$, and thus $l(G) \leq \frac{(n-1)(k-1)}{2} + d(x) \leq \frac{n(k-1)}{2}$. □

Now these results are *extremal results* in the sense that how large G can be without some subgraph. But what about K_n?

Definition. Say G is k-partite, with vertex classes $V_1 \cdots V_k$. If $V_1, \cdots V_k$ partition V and $G[V_i]$ is empty for all i, then G is complete k-partite if:

$$E(G) = \{xy : x \in V_i, y \in V_j, i \neq j\}$$

The *Turan Graph* $T_k(n)$ is the complete k-partite graph on vertex classes $V_1, \cdots V_k$, where $|V_1|, \cdots |V_k|$ are as equal as possible.

Note. - $T_{r-1}(n)$ has no K_r, because it is $(r-1)$-partite.

- $T_{r-1}(n)$ is maximal K_r-free: add any edge creates a K_r.

- To obtain $T_k(n)$ from $T_k(n-1)$, add a point to the smallest class.

Theorem. Let G have n vertices,and $e(G) = e(T_{r-1}(n))$, $G \not\supset K_r$, then $G = T(n)$. In particular, the maximal graph on n vertices without containing a K_r is the $T_{r-1}(n)$ graph.

Proof.

Claim. $\delta(G) \leq \delta(T(n))$.

Proof. Now we have $\sum d_G(x) = \sum d_{T(n)}(y)$ by condition, but we know that $d_{T(n)}(y)$ are as equal as possible, so the smallest value of $d_G(x)$ must be smaller or equal to the smallest value of $d_{T(n)}(y)$, so $\delta(G) \leq \delta(T(n))$. $\square$

Now we induct on n, and $n \leq r-1$ is trivial as $T(n) = K_n$. Now choose $x \in G$ of minimum degree, and let $G' = G - x$. $|G'| = k-1$, $G' \not\sup K_r$, and $e(G') = e(G) - \delta(G) \geq e(T(n)) - \delta(T(n)) = e(T(n-1))$. Then by induction, $G' = T(n-1)$, and $\delta(G) = \delta(T(n))$. Let's say G' has vertex classes $V_1, \cdots, V_{r-1}$. Now $P_G(x)$ cannot meet every V_i, as otherwise, we have $G \supset K_r$, but $d_G(x) = k - 1 - \min_i |V_i|$. Therefore, the only class that x does not meet can only be the smallest class, so if we add x back, by properties of the Turan graph, we must have $G = T(n)$, and thus we are done. $\square$

4.1 The Problem of Zarankiewicz

How many edges can a bipartite graph with each class of size n have if it does not contain s $K_{t,t}$? Let the maxima graph be $Z(n,t)$¿

Theorem. Let $t \geq 2$, then $Z(n,t) \leq 2n^{2-\frac{1}{t}}$ for n sufficiently large.

Proof. Let G have vertex classes X, Y, $|X| = |Y| = n$, $G \not\supset K_{t,t}$. Without loss of generality $d(x) \geq t-1$ for all $x \in X$, else we add an edge. now let d be the average defree of all $x \in X$. Then for a fixed $A \subset Y$, $|A| = t$, we must have $\leq t-1$ $x \in X$ that is connected to it by definition as it does not contain a $K_{t,t}$. So the number of (x, A) with $x \in X$, $A \subset Y$,$|A| = t$, $P(x) > A$ is $\leq \binom{n}{t}(t-1)$. But this number is exactly $\sum_{x \in X} \binom{d(n)}{t}$. And since $\binom{n}{t}$ is a convex function, we have

$$n\binom{n}{t} \leq \sum_{x \in X} \binom{d(n)}{t} \leq \binom{n}{t}(t-1)$$

Then we approximate and expand the two sides:

$$\frac{n(d-t+1)^n}{t!} \leq \frac{n^t}{t!}(t-1)$$

And after rearranging we have $d \leq (t-1)^{\frac{1}{t}} n^{1-\frac{1}{t}} + t - 1$. And thus when n is large, we have $nd \leq 2n^{2-\frac{1}{t}}$ as required. □

Now we know that $\frac{l(G)}{\binom{k}{2}} > 1 - \frac{1}{r-1} \Rightarrow G > K_r$. The following theorem generalizes this result, writing $K_r(t) = T_r(rt)$:

Theorem.

$$\frac{l(G)}{\binom{n}{2}} > 1 - \frac{1}{r-1} + \epsilon \Rightarrow G > K_r(t)$$

for all ϵ and for n large.

5 Colorings

Definition. A *k-coloring* of graph G is a function $c : V \to \{1, \cdots, k\}$ such that $c(x) \neq c(y)$ for all $xy \in G$. The *chromatic number* $\chi(G)$ is the least k such that G is *k-colorable.*

Thus by definition ($c(x) \neq c(y)$) that G k-colorable $\Leftrightarrow$ G is k-partite.

Example.

- $\chi(P_k) = 2$
- $\chi(C_n) = \begin{cases} 2 & \text{if n even} \\ 3 & \text{if n odd} \end{cases}$
- $\chi(K_n) = n$
- $\chi(E_n) = 1$
- $\chi(K_{m,n}) = 2$.

We have a relatively easy proposition:

Proposition. Let G have maximum degree Δ. Then $\chi(G) \leq \Delta + 1$.

Proof. We color in $\Delta+1$ colors, and when we come to color x_i, we have $\leq d(x_i) \leq \Delta$ colors already used on neighbours of x_i, so we have enough colors to color it. □

Now we can see through K_r that this is the "best bound" in terms of degree.

5.1 Coloring Planar Graphs

Theorem. Every Planar Graph is 6-colorable.

Proof. We induct on n. This is trivially true for $N \leq 6$. Given G planar, $n > 6$, we have $e(G) = 3n - 6$, so $\sum d(x_i) \leq 6n - 12$, thus there exist x for degree ≤ 5. Then $G - x$ is 6-colorable, and thus we can color x, so we are done. □

Theorem (Five Color Theorem)**.** If G is planar, G is 5-colorable.

Proof. We use induction on $|G|$ and it is trivial for $|G| \leq 5$. Now given planar G, $|G| > 5$, we have $\delta(G) \leq 5$ as before, so we delete that vertex and color $G - x$ with 5 colors. Then we add x back in, and we are done unless $d(x) = 5$ and all 5 colors appear in $N(x)$. Say the neighbourhood is $x_1, \cdots, x_5$ with x_i having color i for each i. Now we look at a $1 - 3$ path P, where this means a path with alternating color 1 and color 3. we have two cases:

$x_3 \notin P$ Then let H be the *1-3 component* of x_1. Swap all colors from 1 to 3 on H. This is still a valid coloring, but now we have x_1 of color 3, so we use 1 for x.

$x_3 \in P$ Then we cannot have a 2-4 path from x_2 to x_4 as it would have to meet P since G is planar. So we finish by doing a $2 - 4$ swap as above.

□

Proposition. Let G be a connected graph, not complete or an odd cycle. Then $\chi(G) \leq \Delta$, where Δ is the maximum degree in G.

Remark. If G is connected and not regular, we can certainly color it with Δ colors by finding a vertex with degree $< \Delta$, finding edges to connect all other vertices, run greedy backwards. All vertices before the last one has a forward edge, meaning one that has not been colored yet, so Δ colors is enough. And the last one has degree $< \Delta$, so this is enough also.

Proof. Wlog G is Δ- regular (by remark), with $\Delta) \geq 3$. We may assume G is 2-connected, as if x is a cutvertex then let $G_1, \cdots, G_k$ be the components of $G - x$ together with x. Then each G_i is $\Delta-$-colorable, as for every graph with one vertex less than degree Δ, it is Δ colorable.

G is 3-connected

Now choose $x_n \in G$. Must have some $x_1, x_2 \in \Gamma(x_n)$ with $x_1 x_2 \notin E$, else the neighbourhood of x_n and x_n forms a $K_{\Delta+1}$, which is not G as G is not complete, so there wxists a vertex in $K_{\Delta+1}$ that conencts to something not inside that $K_{\Delta+1}$, which contradicts the maximal degree constraint. Now $G - (x_1, x_2)$ is connected, so we can order it as $x_1, \cdots, x_n$ so that each x_i with $3 \leq i \leq n - 1$ has a forward edge. And we run greedy.

G is not 3-connected

Let $G_1, \cdots G_k$ be components of $G-(x,y)$ together with x,y. Then each G_i has a Δ-coloring. If $xy \in E$ then each C_i gives x,y different colors, so we can (recolor and) combine to obtain a Δ-coloring of G. If for each i, we have $d_{G_i}(x) \leq \Delta - 2$ or $d_{G_i}(y) \leq \Delta - 2$, then we are done as we can recolor one of x or y so that it is of two different colors and then we can obtain a Δ coloring again. Now thus for some G_i we have $d_{G_i}(x) = d_{G_i}(y) = \Delta - 1$. And thus $k=2$, with $d_{G_2}(x) = d_{G_2}(y) = 1$. Then let x be connected to u in G_2, and y be connected to v in G_2, Then u, y is a separator and it is not of this form, so we are done. □

5.2 The Chromatic Polynomial

For graph G, let $P_G(t)$ be the number of t-colorings of G. So:

Example.

$P_{K_n}(t) = t(t-1)\cdots(t-n+1)$

$P_{E_n}(t) = t^n$

$P_{P_n}(t) = t(t-1)^n$ [Thus by induction on removing leafs, for tree T on n vertices, $P_T(t) = t(t-1)^{n-1}$]

Given G with an edge $e = xy$, the contraction G/e is obtained from G be replacing vertices x, y by a new vertex e, joined to each z that was adjacent to x or y without multiple edges.

Lemma (Cut-Fuse)**.** Let G be a graph, with an edge e. Then $P_G = P_{G-e} - P_{G/e}$.

Proof. Consider $x, y \in V(G-e)$. Then the number of colorings with x, y different color is exactly the number of colorings in G. And the number of colorings with x, y same color is exactly the number of colorings in G/e. So we are done. □

Proposition. Let g be a graph on n vertices with m edges. Then the first two terms of $P_G(t)$ are $t^n - mt^{n-1}$.

Proof. We induct on m. It is trivially true for $m=0$. Now given G with $m > 0$ edges, and then we choose edge e. Then $P_{G-e}(t) = t^n - (m-1)t^{n-1} + \cdots$ by induction, and $P_{G/e}(t) = t^{n-1} - \cdots$ by induction. So by Cut-Fuse, we are done. □

Remark.

- Since $P_G$is a polynomial, we can talk about $P_G(t)$ for t real (or complex).
- Historically, P_G is introduced to try to prove the 4-color theorem by saying that P_G does not have a root at $t = 4$ for planar G. No such algebraic proof of this is known but it is known that $P_G(\phi + 2) \neq 0$, where $\phi = \frac{1+\sqrt{5}}{2}$.

5.3 Edge-Colorings

A *k-edge-coloring* of G is a function $C : E(G) \to \{1, \cdots, n\}$ such that $c(e) \neq c(f)$ when e, f are adjacent edges. The *edge-adjacent number* or *chromatic index* of G is the least k such that G has a k-edge coloting, written $\chi'(G)$.

Theorem (Vizing's Theorem)**.** For any graph G, $\chi'(G) = \Delta$ or $\Delta + 1$.

Proof. Now induct on $e(G)$, $e(G) = 0$, given G with $e(G) > 0$, we choose $e = xy_1$, and then we have $\Delta + 1$ coloring of $G - e$. At each vertex, some color is missing as we have $\Delta + 1$ colors. Then we choose maximal sequence as followes: Start by choosing a color C_1 at y_1 that is missing. If x is also missing at x, we are done. If not, then there exist $xy_2 \in E(G)$ with color C_1. Now y_2 is missing color C_2, and we can inductively choose *new* edges xy_{i+1} of color C_i.

As G is finite, this terminates, so either we have no edge from x has C_k, in which we are done by recoloring xy_i with C_i, or $C_k = C_j$ for some $k > j$. Wlog $j = 1$ because we give edge xy_{i+1} color c_i leaving xy_1 un colored. Let C be a color missing at x, and let H be the $C - C_1$ component of X.

$y_1 \notin H$ Swap C, C_1 on H and we have now C_1 missing at x and at y_1, so we can give xy_1 color C_1.

$y_1 \in H, y_k \notin H$ We now swap C, C_1, and color xy_1 C. Note that this is possible as this component ends on y_1 with an edge C as y_1 is missing C_1.

$y_1, y_k \in H$ This is not possible as in this component y_1 has degree 1, y_k has degree 1, x has degree 1, and it is a path or cycle by definition.

□

5.3.1 Graph on Surfaces

Up to now, we have considered graphs in general or on a plane. But why stop there?

Definition. The surface of genus g consists of a sphere with g handles attached, or more colloquially, g donut holes.

We can expand Euler's formula to show that:

Lemma.

$$V - E + F \geq 2 - 2g$$

Using this we have the following important theorem:

Theorem (Heawood's Theorem)**.** Let G be a graph drawn on a surface of Euler characteristic $\mathbb{E} = 2 - 2g \leq 0$. Then

$$\gamma(G) \leq H(E) = \lfloor \frac{t + \sqrt{49 - 24\mathbb{E}}}{2} \rfloor$$

.

Proof. Let G be a graph drawn on the surface with $\chi(G) = k$. We want $k \leq H(\mathbb{E})$. Choose minimal G with $\chi(G) = k$, then $\delta(G) \geq k - 1$ by minimality of G and $n \geq k$.

We know $|E| \leq 3(n - \mathbb{E})$, by Euler formula above, rearranged by using $3F \leq 2E$, so the sum of all degrees is $2|E| \leq 6(n - \mathbb{E})$ and so $\delta(G) \leq 6 - \frac{6E}{n}$.

Thus $k - 1 \leq \delta(G) \leq 6 - 6\frac{E}{n} \leq 6 - 6\frac{E}{k}$ as $E \leq 0$ and $k \leq n$. So we have:

$$k^k - k \leq 6k - 6E \quad \Rightarrow \quad k^2 - 7k + 6E \leq 0$$

This gives the required result. □

Note. Equality does hold! This took 75 years to prove.

6 Ramsey Theory

Definition. We define $R(s,t)$ as the smallest n such that whenever K_n is 2-edge-colored, we either have a red K_s or a blue K_t.

Remark. This seems quite weird. Why do we want to study this? Think of the 2-coloring as disorder while the red and blue subgraphs as order. This is essentially finding order in disorder, which is important in chaos theory.

Theorem. $R(s,t)$ exists for all $s,t \geq 2$. Moreover, $R(s,t) \leq R(s-1,t)+R(s,t-1)$ for all $s,t \geq 3$.

Proof. It is enough to show that $R(s,t) \leq R(s-1,t)+R(s,t-1)$ whenever $R(s-1,t)$ and $R(s,t-1)$ exists, then we are ddone by induction on $s+t$. Let $a = R(s-1,t)$ and $b = R(s,t-1)$. Given a 2-coloring of K_{a+b}, we seek red K_s or blue K_t. Choose $x \in V(K_{a+b})$: $d(x) = a+b-1$ so x incident with $\geq a$ red edges or $\geq b$ blue edges. Wlog $\geq a$ red edges. Consider the K_a spanned by a red neighbours by definition of a. This K_a must contain either a blue K_t or a red K_{s-1} (in which x forms a X_s with these points), so we are done. □

Remark. - Hence for any graph G on n vertices, we have $K_s \subset G$ or $K_s \subset \bar{G}$.

- Equivalently, G contains either K_s or an independent set of s vertices.

Corollary. For $s,t \leq 2$ have $R(s,t) \leq \binom{s+t-2}{s-1}$. In particular, $R(s) \leq 2^{2s}$.

Proof. Induction on $s+t$: We are done if $s=2$ or $t=2$. Given $s,t \geq 3$, we have:

$$R(s,t) \leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}$$

□

For $k \geq 1$, and $S_1 \cdots, S_k \geq 2$, we write $R_k(S_1, \cdots, S_k)$ for least n, if exists, such that whenever $E(K_n)$, $k-$colored and exist K_{S_i} of color i.

Theorem (Ramsey Theorem for k colors)**.** $R_k(S_1, \cdots, S_k)$ exists for all $k \geq 1$ for all $S_1, \cdots S_k \geq 2$.

Proof. We induct on k. Given $k \geq 3$ and $S_1 \cdots S_k \geq 2$, we will show that:

$$R_k(S_1, \cdots, S_k) \leq R(S_1, R_{k-1}(S_2, \cdots, S_k))$$

This is by using the the 2 color Ramsey existence proof, as for $K_n, n = R(S_1, R_{k-1}(S_2, \cdots, S_k)$, we either find a K_{S_1} colored 1, or a K_m, $m = R_{k-1}(S_2, \cdots, S_k)$ and we are done by induction. □

Now let's extend this even further (seriously?). Suppose we color each triangle of K_n red or blue - can we find 4 points, all of whose triangles are colored the same? For an r-set, in which the example above was $r=3$, and $s,t \geq r$, we write $R^{(r)}(s,t)$ for the least n such that whenever $X^{(r)}$ is 2-colored, there exist a red s set or a t blue set.

Example. - $R^{(2)}(s,t) = R(s,t)$.

- $R^{(r)}(s,t) = s$.

Theorem. $R^{(r)}(s,t)$ exists for all $r \geq 1$ and $s,t \geq r$.

Proof. We induct on r. We know $r = 1, 2$ is true. Given $r \geq 3$, induct on $s+t$, in which we can prove the base case $s = r$ and $t = r$. Given $s,t > r$. We'll show that:

$$R^{(r)}(s,t) \leq R^{(r-1)}(R^{(r)}(s-1,t), R^{(r)}(s,t-1))$$

Let $a = R^{(r)}(s-1,t)$, $b = R^{(r)}(s,t-1)$, $n = R^{(r-1)}(a,b) + 1$. Given 2-coloring of $X^{(r)}$. Fix $x \in X$. Then c induces a 2-coloring of the $(r-1)$ sets of $Y = X - x$. By definition of n, this contains a red a-set or a blue b-set, wlog a red a-set Z. So each $r-1$ set $A \subset Z$ has $A \cup \{x\}$ having colot red. By definition of a, Z contains either a blue t-set for c, or a red $(s-1)$ set for c, in which along with x gives a red s-set. □

Now similarly we can do this on k colors. But what about infinity?

6.1 Infinite Ramsey Theory

Theorem. Let $\mathbb{N}^{(2)}$ be the graph consisting of $\mathbb{N}$ vertices and all edges. Then there exists an infinite monochromatic $M \subset \mathbb{N}$.

Proof. Choose $a_1 \in \mathbb{N}$. Then there exist infinite $B_1 \subset \mathbb{N}\ \{a_1\}$ such that all edges from $a_1 \to B_1$ have the same color, say c_1. Choose $a_1 \in B_1$. Then there exists infinite $B_2 \subset B_1\{a_2\}$ such that all edges from a_2 to B_2 are the same color. Continue inductively. Then we obtain $a_1, \cdots$, in $\mathbb{N}$ where $a_i a_j$ has color c_i for $i < j$. Then we must have infinitely many of them having the same color as there are only 2 colors, then that subset is infinite monochromatic by construction. □

Remark. This can be similarly done for k colors, and this type of proof is called a *2-pass* proof.

We will now do the same for r-sets.

Theorem (Infinite Ramsey for r-sets)**.** Let $r \geq 1$, and let $\mathbb{N}^{(r)}$ be 2-colored. Then there exists an infinite monochromatic subset $M \subset \mathbb{N}$.

Proof. Induct on r and we know $r = 2$ is true. Given $r > 2$, and a 2-coloring c of $\mathbb{N}^{(r)}$, choose $a_1 \in \mathbb{N}$. This infuces a 2-coloring c' of $\{\mathbb{N}\ \{a_1\}\}^{(r-1)}$ by $c'(F) = c(F \cup \{a_1\})$. So there exist an infinite monochromatic $B_1 \subset \mathbb{N}\backslash\{a_1\}$ for c', by the induction hypothesis. So we have some c_1 such that every r-set of the form $\{a_1 \cup F\}$ for $F \subset B_1$ has color c_1.

Then we can similarly do this inductively and obtain points $a_1, \cdots \in \mathbb{N}$ and colors c_1, c_2, such that any r-set $a_{i_1} \cdots a_{i_r}$ has color c_{i_1}. but infinitely many of the c_i agree, so that subset is monochromatic. □

7 Random Graphs

Theorem. let $s \geq 3$. Then $R(s) > 2^{n/2}$.

Proof. Choose a coloring of K_n at random, by taking each edge to be red or blue with probability 0.5 each. Then the probability that a fixed s-set is monochromatic os $2(\frac{1}{2})^{\binom{s}{2}}$. The number of s-sets is $\binom{n}{s}$, so the probability that there exist a monochromatic s set is less than $\binom{n}{s}e^{1-\binom{s}{2}}$. Thus we must have $R(s) > n$ if $\binom{n}{s}2^{1-\binom{s}{2}} < 1$, or when $\binom{n}{s} < 2^{\binom{s}{2}-1}$. But $\binom{n}{s} \leq \frac{n^s}{s!}$ and $s! \geq 2^{s/2+1}$, so the equation becomes:

$$n^2 \leq 2^{s^2/2} \Rightarrow n \leq 2^{s/2}$$

□

Remark.
- This is a *random graphs* argument.
- This gives no hint on how to construct such a coloring, but no construction giving even an exponential lower bound on $R(s)$ is known!

The *probability space* $G(n,p)$ is defined on the set of all graphs in vertex set $\{1, 2, \cdots, n\}$ by choosing each edge to be present with probability p independently.
Recall Zarankiewicz, and we had $Z(n,t) \leq 2n^{2-1/t}$. Now we would use random graph theory to prove a lower bound:

Theorem. $Z(n,t) > \frac{1}{2}n^{2-\frac{2}{t+1}}$.

The idea is that if the graph has m edges and r copies of $k_{t,t}$, then we remove an edge from each copy of $K_{t,t}$ to obtain a graph with $\geq m - r$ edges and no $K_{t,t}$ showing that $Z(n,t) > m - r$.

Proof. Choose a random bipartite graph G, vertex classes X, Y by taking each edge independently with probability p. Let $M = e(G)$, and R be the number of $K_{t,t}$ in G. Now the number of $K_{t,t}$ is $\binom{n}{t}^2$, and the probability that each one exist is p^{t^2}. So $\mathbb{E}(R) = \binom{n}{t}^2 p^{t^2}$ and $\mathbb{E}(M) = pn^2$. Using linearity of expectation:

$$\mathbb{E}(M - R) = pn^2 - \binom{n}{t}^2 p^{t^2} \geq pn^2 - \frac{1}{2}n^{2t}p^{t^2}$$

Now we want to maximize this. Choose *magic* $p = n^{-2/(t+1)}$, and we have $\mathbb{E}(M - R) \geq \frac{1}{2}n^{2-2/(t+1)}$, so there exist a graph with $m - r \geq \frac{1}{2}n^{2-2/(t+1)}$, so we are done. □

7.1 Graphs with large $\chi(G)$

What does a large $\chi(G)$ graph intuiteively look like? We would want a lot of edges to be connected to a lot of points, so that the colors are very much constrained and thus you need more of them. But it turns out that one has the mind-blowing theorem below:

Theorem. For all k, g, there exists a graph G with $\chi(G) \geq k$ and girth(G)$\geq g$.

Remark. We try to find G on n vertices such that the number of short cycles of length $\leq g$ is less than $n/2$, abd every independent set has size $\leq n/2k$. Then we are done by removing a vertex in each short cycle to obtain H with girth $\geq g$, and $\chi(H) \geq$

$(n/2)/(n/2k) = k$. The latter result comes from the fact that each monochromatic subset in a $\chi(H)$ coloring is an independent set, so $a(G)\chi(G) \geq |G|$, where $a(G)$ is the maximum size of an independent subset.

Proof. We choose a random G with $p = n^{-1+1/g}$. Then let x_i be the number of i-cucles in G, $3 \leq i \leq g-1$, and let x be the number of cycles of length $< g$, so $x = x_3 + \cdots, +x_{g-1}$. Now

$$E(x_i) \leq n^i p^i$$

So

$$E(x) \leq \sum_{i}^{g-1} (np)^i \leq \sum n^{i/g} \leq gn^{(g-1)/g} = n\frac{g}{n^{1/g}} < \frac{n}{4}$$

When n is large. Then $P(x \leq n/2) > \frac{1}{2}$. Let $t = \frac{n}{2k}$ for n a multiple of $2k$ and let y be the number of t-sets that are independent. There are $\binom{n}{t}$ t-sets and the probbaility that a given t set is independent is $(1-p)^{\binom{t}{2}} \leq e^{-p\binom{t}{2}}$. So:

$$E(y) \leq n^t e^{-p\binom{t}{2}} \leq e^{\frac{n}{2k}\log n - \frac{n^2}{8k^2}n^{-1+1/g}} \to 0$$

As $n \to \infty$ as the second term is asymptotically larger. So for n large, $E(y) < \frac{1}{2}$, so $P(y = 0) \geq \frac{1}{2}$. Then there exists G with $x \leq \frac{n}{2}$, and $y = 0$, so we are done. □

7.2 Structure of Random Graphs

What does a typical G look like? How do its properties change as p varies?

7.2.1 Probability Reminder

If we want to show that $P(X = 0)$ is large, for X an integer random variable, we only need to show that μ is really small, $\mu < 1 \Rightarrow P(X = 0) \geq 1 - \mu$.

Now if we want to show that $P(X = 0)$ is small, then using Chebeshev's inequality $P(|X - \mu| > t) \leq \frac{V}{t^2}$, and taking $t = \mu$, we have:

$$P(X = 0) \leq P(|x - \mu| \geq \mu) \leq \frac{V}{\mu^2}$$

Now we assume X counts the number of some events A that occur: $\mu = E(X) = \sum_A P(A)$. Also $E(X)^2 = \sum_{A,B} P(A)P(B)$. and $E(X^2) = \sum_{A,B} P(AB)$. By independence of A and b:

$$V = \sum_A P(A) \sum_B (P(B|A) - P(B))$$

Theorem. Let λ be fixed, and choose random $G = G(n,p)$ where $P = \lambda\frac{\log n}{n}$, then if $\lambda < 1$ almost surely G has an isolated vertex, and if $\lambda > 1$ almost surely G has no isolated vertex.

Proof. Let X be the number of isolated vertices of G. Then:

$$\mu = E(X) = n(1-p)^{n-1} = \frac{n}{1-p}(1-p)^n$$

For $\lambda > 1$, we have $\mu \leq \frac{n}{1-p}e^{-pn} = \frac{n}{1-p}e^{-\lambda \log n} = \frac{n^{1-\lambda}}{1-p} \to 0$ as $n \to \infty$. So certainty we have $P(X = 0) \to 1$.
For $\lambda < 1$,we have $1 - p \geq e^{-(1+\delta)p}$ for any δ. So:

$$\mu \geq \frac{n}{1-p}e^{-(1+\delta)pn} = \frac{n}{1-p}e^{-(1+\delta)\lambda \log n} = \frac{n^{1-(1+\delta)p}}{1-p}$$

Choosing δ such that $(1 + \delta)\lambda < 1$, we have $\mu \to \infty$ as $n \to \infty$.

$$\begin{aligned}\text{Var}(x) &= \underbrace{n(1-p)^{n-1}(1-(1-p)^{n-1})}_{N \text{ terms where } A=B} + \underbrace{n(n-1)(1-p)^{n-1}((1-p)^{n-2}-(1-p)^{n-1})}_{n(n-1) \text{ terms where } A \neq B} \\ &= \mu + n(n-1)(1-p)^{n-1}p(1-p)^{n-2} \\ &= \mu + \frac{p}{1-p}n^2(1-p)^{2n-2} = \mu + \frac{p}{1-p}\mu^2\end{aligned}$$

Therefore $\frac{V}{\mu^2} \to 0$ as $n \to \infty$. So $P(X = 0) \to 0$ as $n \to \infty$. □

Theorem. Let $0 < p < 1$ be fixed, and let d be a real number with $\binom{n}{d}p^{\binom{d}{2}} = 1$. Then almost surely, $G(n, p)$ has clique number equal to $\lceil d \rceil$, $\lfloor d \rfloor$, or $\lfloor d \rfloor - 1$.

Sketch. Fix integer k and let X be the number of K_k in G. So $\mu = \binom{n}{k}p^{\binom{k}{2}}$. Check that for $k \geq d + 1$, we have $\mu \to 0$ and for $k \leq d - 1$, we have $\mu \to \infty$. Now:

$$V = \binom{n}{k}p^{\binom{k}{2}}\sum_{s=2}^{k}\binom{k}{s}\binom{n-k}{k-s}(p^{\binom{k}{2}-\binom{s}{2}} - p^{\binom{k}{2}})$$

Then we have $\frac{V}{\mu^2} \leq \frac{1}{\mu}\sum_{s=2}^{k}\binom{k}{s}\binom{n-k}{k-s}p^{\binom{k}{2}-\binom{s}{2}}$. We can check that the first and last term dominates, and bound this by something that goes to 0 as $n \to \infty$. Then we are done. □

8 Algebraic Methods

Definition. The diameter of a graph is defined as $\max d(x, y) : x, y \in G$. So if G has diameter 1 then G is complete.

What about diameter 2? How many vertices can G have if G has diameter 2 and maximum degree Δ? If we look from a point x, we must have:

$$V(G) = \{x\} \cup \{N(G)\} \cup \{N(N(G))\}$$

Since $|N(G)| \leq \Delta$, we have $|G| \leq 1 + \Delta^2$. And it is easy to see that if equalitt is achieved then G is just Δ regular. We call this a Δ*-regular Moore graph.*
Thus, the big question is:

For what Δ do we have Moore graphs?

To do this, we introduce algebraic methods:

Definition. The *adjacency matrix* of G is what you think it is: It is an $|G| \times |G|$ matrix with $A_{ij} = 1$ if $ih \in E(G)$ and 0 otherwise.

Why do we define this? Because matrices are nice:

Remark. $(A^N)_{ij}$ describes the number paths from i to j of length N. For $N = 1$ this is trivially true and one can see higher powers through expanding matrix multiplication.

Moreover, this matrix is real symmetric, and thus it has $|G|$ eigenvalues, with the following properties:

Proposition. For λ an eigenvalue and Δ/δ the maximum/minimum degree of G:

(i) $\lambda \leq \Delta$

(ii) For G connected: $-\Delta$ is an eigenvalue $\Leftrightarrow$ G is regular and bipartite.

(iii) $\max \lambda \geq \delta$.

Proof. (i) Choose eigenvector x for λ, and choose i with $|x_i|$ maximal. Wlog $x_i = 1$ by rescaling, then $|(Ax)_i| = |\sum_{j \in N(i)} x_j| = \lambda \leq \Delta$. So done.

(ii) $\Rightarrow$ From (i) we see that we must have $d(i) = \Delta$ and $x_j = 1$ for all $j \in d(i)$. Then we can repetitively do this on each $j \in N(i)$. Then $d(k) = \Delta$ for all $k \in G$ as G is connected.

$\Leftarrow$ Just let $x = (1, \cdots, 1)$ Then $Ax = (\Delta, \cdots, \Delta)$.

(iii) $\Rightarrow$ Similarly from (i) we must have $d(j) = \Delta$ and $x_j = -1$ for all $j \in N(i)$. Then we can repeat and discover that for $ij \in E(G)$, $x_i, x_j = 1, -1$ or the reverse order, and thus we can separate the graph into two classes, one only having 1s and one only having -1s.

$\Leftarrow$ Let X, Y be the two vertex classes and choose $x = (\underbrace{1, 1, \cdots, 1}_{\text{X}}, \underbrace{-1, -1 \cdots, -1}_{\text{Y}})$.

(iv) Let $x = (1, 1 \cdots, 1)$. Then $(Ax)_i \geq \delta$ for all i, so $(Ax, x) \geq \delta(x, x) = \delta n$. But by eigenvector decomposition $\min(Ax, x) = \lambda_{min}|x|^2$, so $\lambda_{min} \geq \delta$.

□

We now have a strengthening of a previous result:

Proposition. For any graph G, $\chi(G) \leq \lambda_{max} + 1$.

Proof. We induct on $|G|$. Clearly done if $|G| = 1$. Given G with $|G| > 1$, we choose $v \in G$ with minimal degree $\leq \lambda_{max}$. Then let B be the adjacency matrix of $G - v$ and wlog we removed the last row and column. Then for any $x = (x_1, \cdots, x_{n-1})$, we have $y = (y_1, \cdots, y_{n-1}, 0)$. Then $(Ay, y) = (Bx, x)$, and similarly by eigenvector decomposition $(Bx, x) = (Ay, y) \leq \max_\lambda G|x|^2$, so $\max_\lambda G - v \leq \max_\lambda G = \lambda_{max}$. So we can color $G - v$ by the induction step, and v since $d(\lambda_{max}) \leq \lambda_{max}$. □

8.1 Moore Graphs

Wait, we are getting distracted. What about Moore graphs? Don't worry, you have the section heading, but some more definitions first.

Definition. A graph G is *strongly regular* with parameters (k, a, b) if G is k-regular, any two adjacent points have exactly a common neighbours, and any two non-adjacent points having exactly b common neighbours.

If we look back to how we defined Moore graphs, we would see that a degree k Moore graph is strongly regular with $(k, 0, 1)$. Here comes the machinery:

Proposition (Rationality condition for strongly regular graphs)**.** Let G be a graph on n vertices and is strongly regular with (k, a, b), and $b \geq 1$ so that G is connected. Then:

$$\frac{1}{2}\left(n - 1 \pm \frac{(n-1)(b-a) - 2k}{\sqrt{(a-b)^2 + 4(k-b)}}\right)$$

are integers.

Proof. G is connected, so k is an eigenvalue with multiplicity 1. What are the other eigenvalues? We have that:

$$A^2_{ij} = \begin{cases} k & \text{if } i = j \\ a & \text{if } i \neq j, ij \in E \\ b & \text{if } i \neq j, ij \notin E \end{cases}$$

Thus we have $A^2 = kI + aA + b(J - I - A)$ where J has all entries equal to 1. For an eigenvalue λ that is not k, since k's eigenvector is $(1, \cdots, 1)$, the eigenvector of λ, x, is orthogonal to that, so $Jx = 0$. Then:

$$(\lambda^2 + (b-a)\lambda + (b-k))x = 0$$

Thus we have $\lambda^2 + (b-a)\lambda + (b-k) = 0$. So eigenvalues not equal to k are:

$$\lambda = \frac{1}{2}(a - b \pm \sqrt{(b-a)^2 + 4(k-b)})$$

Denote them by λ, μ, and their multiplicities r, s. Then $r + s = n - 1$ by deinfition, and since the eigenvalues sum up to 0 as A has trace 0, we also have $\lambda r + \mu s = -k$. Solving for r, s gives the result above. ☐

Finally, finally, we are here:

Corollary. If there exists a Moore graph of degree k, then $k \in \{2, 3, 5, 57\}$.

Proof. Now we know $n = k^2 + 1, a = 0, b = 1$, so we must have:

$$\frac{1}{2}\left(k^2 \pm \frac{k^2 - 2k}{\sqrt{1 + 4(k-1)}}\right) \in \mathbb{Z}$$

So we either have $k^2 - 2k = 0$ or $4k - 3$ is a perfect square. If $k^2 - 2k = 0$ then $k = 2$. If $4k - 3 = t^2$, then t divides:

$$k^2 - 2k = \left(\frac{t^2+3}{4}\right)^2 - 2\left(\frac{t^2+3}{4}\right)$$

So t divides $t^4 - 2t^2 + 15$. So t divides 15, which gives $t = 1, 3, 5, 15$, or $k = 1, 3, 7, 57$. $k = 1$ is not possible so we are done. □

Remark. $k = 2$ is the C_5 graph, $k = 3$ is the Petersen graph, $k = 7$ is called the *Hoffman-Singleton Graph*, but for $k = 57$ we still have no idea if such a graph exists.