Fluid Dynamics Review Sheet

Michael Li. Graphs: Dexter Chua

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Parallel viscous flow

Plane Couette flow, dynamic viscosity. Momentum equation and boundary conditions. Steady flows including Poiseuille flow in a channel. Unsteady flows, kinematic viscosity, brief description of viscous boundary layers (skin depth). [3]

Kinematics

Material time derivative. Conservation of mass and the kinematic boundary condition. Incompressibility; streamfunction for two-dimensional flow. Streamlines and path lines. [2]

Dynamics

Statement of Navier-Stokes momentum equation. Reynolds number. Stagnation-point flow; discussion of viscous boundary layer and pressure field. Conservation of momentum; Euler momentum equation. Bernoulli's equation.

Vorticity, vorticity equation, vortex line stretching, irrotational flow remains irrotational. [4]

Potential flows

Velocity potential; Laplace's equation, examples of solutions in spherical and cylindrical geometry by separation of variables. Translating sphere. Lift on a cylinder with circulation.

Expression for pressure in time-dependent potential flows with potential forces. Oscillations in a manometer and of a bubble. [3]

Geophysical flows

Linear water waves: dispersion relation, deep and shallow water, standing waves in a container, Rayleigh-Taylor instability.

Euler equations in a rotating frame. Steady geostrophic flow, pressure as streamfunction. Motion in a shallow layer, hydrostatic assumption, modified continuity equation. Conservation of potential vorticity, Rossby radius of deformation. [4]

Contents

1 Parallel viscous flow

1.0 Preliminaries

Definition (Fluid)**.** A *fluid* is a material that flows.

Example. Air, water and oil are fluids. These are known as *simple* or *Newtonian* fluids, because they are simple. And these are the only fluids that we would dare care about in this course.

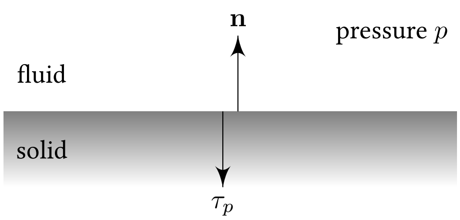
Definition (Newtonian fluids and viscosity)**.** A *Newtonian fluid* is a fluid with a linear relationship between stress and rate of strain. The constant of proportionality is *viscosity*.

Definition (Stress)**.** *Stress* is force per unit area. Example: Pressure.

Definition (Strain)**.** *Strain* is the extension per unit length. The *rate of strain* is $\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{strain})$ is concerned with gradients of velocity.

1.1 Stress

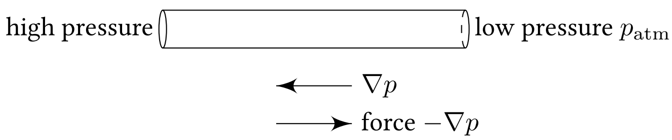
Suppose we have a fluid with pressure p acting on a surface with unit normal $\mathbf{n}$, pointing *into* the fluid. This causes



Definition (Normal stress)**.** The *normal stress* is

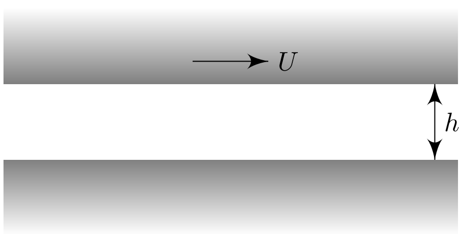
$$\tau_p = -p\mathbf{n}.$$

Note. Pressure acts in emphall directions so there are no directions. A force only arises when we have a change in pressure.



Then this gives a *body force* that drives the water from left to right.

We can also have stress in the horizontal direction. Suppose we have two infinite plates with fluid in the middle. We keep the bottom plane at rest, and move the top plate with velocity U.



By definition, the stress is the force per unit area. In this case, it is the horizontal force we need to exert to keep the top plate moving.

Definition (Tangential stress)**.** The *tangential stress* τ_s is the force (per unit area) required to move the top plate at speed U.

Law. For a Newtonian fluid, we have

$$\tau_s = \mu \frac{\partial u_T(\mathbf{x})}{\partial \mathbf{n}}.$$

Where μ is the dynamic viscosity.

1.2 Steady parallel viscous flow

We are first going to consider a very simple type of flow, known as *steady parallel viscous flow*, and derive the equations of motion of it. We first explain what this name means. The word "viscous" simply means we do not assume the viscosity vanishes, and is not something new.

Definition (Steady flow)**.** A *steady flow* is a flow that does not change in time. In other words, all forces balance, and there is no acceleration.

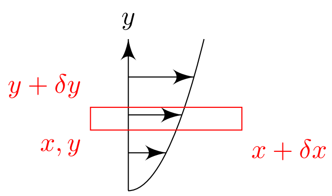
Definition (Parallel flow)**.** A *parallel flow* is a flow where the fluid only flows in one dimension (say the x direction), and only depends on the direction perpendicular to a plane (say the $x - z$ plane). So the velocity can be written as

$$\mathbf{u} = (u(y), 0, 0).$$

These can be conveniently thought of as being two-dimensional, by forgetting the z direction.

The velocity also does not depend on the x direction, as we assume it is imcompressible ($\nabla \dot{x} = 0$).

To derive the equations of motion, we can consider a small box in the fluid.



We know that this block of fluid moves in the x direction without acceleration. So the total forces of the surrounding environment on the box should vanish.

We first consider the x direction. There are normal stresses at the sides, and tangential stresses at the top and bottom. The sum of forces in the x-direction (per unit transverse width) gives

$$p(x)\delta y - p(x + \delta x)\delta y + \tau_s(y + \delta y)\delta x + \tau_s(y)\delta x = 0.$$

By the definition of τ_s, we can write

$$\tau_s(y + \delta y) = \mu \frac{\partial u}{\partial y}(y + \delta y), \quad \tau_s(y) = -\mu \frac{\partial u}{\partial y}(y),$$

where the different signs come from the different normals.

Note. A convenient or helpful way to think about it is because the force is pushing the block on above (faster) and dragging it down below. Although the fluid flows in the same direction, the forces are in different directions.

Dividing by $\delta x \delta y$, we get

$$\frac{1}{\delta x}(p(x) - p(x + \delta x)) + \mu \frac{1}{\delta y}\left(\frac{\partial u}{\partial y}(y + \delta y) - \frac{\partial u}{\partial y}(y)\right).$$

Taking the limit as $\delta x, \delta y \to 0$, we end up with the equation of motion

$$-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} = 0.$$

Performing similar calculations in the y direction, we obtain

$$\frac{\partial p}{\partial y} = 0.$$

Now we allow

$$\mathbf{u} = (u(y, t), 0, 0).$$

Writing the external body force (per unit volume) as $(f_x, f_y, 0)$, we can similarly obtain

$$\begin{aligned} \rho \frac{\partial u}{\partial t} &= -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + f_x \\ 0 &= -\frac{\partial p}{\partial y} + f_y. \end{aligned}$$

Note. To solve this equation, we need the following boundary conditions, the first one used in solid-liquid boundaries, and the second one mainly for fluid-fluid ones:

(i) *No-slip condition*: at the boundary, the tangential component of the fluid velocity equals the tangential velocity of boundary.

$$u_T = 0.$$

(ii) *Stress condition*: If a tangential stress is applied, then we have:

$$-\mu \frac{\partial u_T}{\partial n} = \tau.$$

There are two main important examples:

Couette Flow This is a steady flow with upper boundary moving at U. Assuming this is steady and there is no pressure gradient, our equations give

$$\frac{\partial^2 u}{\partial y^2} = 0.$$

Moreover, the no-slip condition says $u = 0$ on $y = 0$; $u = U$ on $y = h$. The solution is thus $u = \frac{Uy}{h}$.

Poiseuille Flow Now we have a pressure gradient but no moving boundary. Let's assume the pressure gradient is δP per unit length. Our equations give:

$$-\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} = 0$$
$$-\frac{\partial p}{\partial y} - g\rho = 0$$

With $u = 0$ at $y = 0$ and $y = h$.

The boundary conditions are $u = 0$ at $y = 0, h$. We can solve this easily to find

$$u = \frac{\delta P}{2\mu} y(h - y).$$

Here the velocity is the greatest at the middle, where $y = \frac{h}{2}$.

1.3 Properties

Here are some important terms to remember for the course:

Volume Flux The *volume flux* is the volume of fluid traversing a cross-section per unit time. This is given by

$$q = \int_0^h u(y)\,\mathrm{d}y$$

per unit transverse width.

Vorticity Think of this as a measure of rotation or angular momentum. It is defined as $\omega = \nabla \times \mathbf{u}$.

Kinematic Viscosity This is defined as $\nu = \frac{\mu}{\rho}$. Why? We would see in the example below.

Example. Consider fluid initially at rest in $y > 0$, resting on a flat surface.

At time $t = 0$, the boundary $y = 0$ starts to move at constant speed U. There is no force and no pressure gradient.

We use the x-momentum equation to get

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2},$$

where $\nu = \frac{\mu}{\rho}$. This is clearly the diffusion equation, with the diffusivity ν. We can view this as the diffusion coefficient for motion/momentum/vorticity. Now from IA Differential equations, we know (not really since must have forgotten it by now) that this diffusion equation admits a similarity solutionof the form $Uf(\eta)$ where $\eta = \frac{y}{\sqrt{\nu t}}$. Then substitute it in to reach:

$$u = U\,\mathrm{erfc}\left(\frac{y}{2\sqrt{\eta t}}\right).$$

2 Kinematics

So we would now move away from parallel flow and into the more general flow category, but with viscosity removed and no compressibility.

Definition (Material derivative)**.** If $\mathbf{x}(t)$ is the (Lagrangian) path followed by a fluid particle, then necessarily $\dot{\mathbf{x}}(t) = \mathbf{u}$ by definition. In this case, we write

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f$$

Where the last inequality comes from chain rule.

2.1 Equations and How They Got Here

2.1.1 Conservation of Mass

Now the flux through any surface must be 0, because we are assuming the fluid is not compressible. So we have:

$$\int_{\partial \mathcal{D}} \rho \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S = \int \nabla \cdot \mathbf{u} \, \mathrm{d}V = 0$$

For any volume V. So we have $\nabla \cdot \mathbf{u} = 0$. In general, we have:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

As the rate of change of the density must equal to the flux floating out, which by the divergence theorem, is the divergence inside a fluid.

Note. Yes. This is same as the charge conservation.

For solid boundaries, we cannot let fluid pass, so we have a boundary condition. Assume the fluid is moving with speed $\mathbf{U}$. Then switching to a comoving frame we have $\mathbf{u} \cdot \mathbf{n} = 0$, so in the original frame we have $\mathbf{u} \cdot \mathbf{n} = \mathbf{U} \cdot \mathbf{n}$.

2.2 Streamfunction, or Potentials

Since we know our fluid is incompressible, namely $\nabla \cdot \mathbf{u} = 0$, then we have:

Definition (Vector potential)**.** A *vector potential* is an $\mathbf{A}$ such that

$$\mathbf{u} = \nabla \times \mathbf{A}.$$

For a two-dimensional flow, we have $\mathbf{A} = (0, 0, \psi)$, where ψ is called the *streamfunction.*

Note. Important things to know about the streamfunction:

- The fluid flows perpendicular tot he normal of the streamfunction. That is, tangent to its contours.
- The streamfunction is an *instantaneous* picture of the flow *only.*
- The volume flux passing between two points is $\int_{x_0}^{x_1} \mathbf{u} \cdot \mathbf{n} \mathrm{d}l = \int_{x_0}^{x_1} \mathbf{u} \cdot (\mathrm{d}x, \mathrm{d}y) = \psi(x_0) - \psi(x_1)$. So if streamlines (contours of ψ) are closer, the speed is higher, and the volume flux does not change.

3 Dynamics

3.1 Navier-Stokes Equations

Law (Navier-Stokes equation)**.**

$$\rho \frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}\,t} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}.$$

This is basically restating $F = ma$: Mass (ρ) times acceleration is the sum of pressure force, viscosity force and bofy forces $\mathbf{f}$. For this course, the only body force is gravity, so we have $\mathbf{f} = \mathbf{g}\rho$.

3.2 Pressure

We consider the hydrostatic pressure. As we have $\mathbf{u} = 0$, Navier-Stokes become $\nabla p = \mathbf{f} = \rho \mathbf{g}$, so we have $p_H = p_0 - gpz$. This $-gpz$ term gives the Archimedes principle:

$$\mathbf{F} = -\int_{\partial \mathcal{D}} p_H \mathbf{n} \cdot dS = -\int_{\mathcal{D}} \nabla p_H \,\mathrm{d}V = -\int_{\mathcal{D}} \mathbf{g}\rho \,\mathrm{d}V = -\mathbf{g}\int_{\mathcal{D}} \rho \,\mathrm{d}V = -M\mathbf{g}$$

Therefore, the buoyant force cancels out with the gravitational force, so in this course we usually do not consider gravity if the density remains constant.

3.3 Reynolds Number

We take the Navier-Stokes equation and look at the scales of terms.

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u},$$

where again $\nu = \frac{\mu}{\rho}$. We are going to estimate the size of these terms. We get

$$\frac{U}{(L/U)} \qquad U \cdot \frac{U}{L} \qquad \frac{1}{\rho}\frac{P}{L} \qquad \nu \frac{U}{L^2}.$$

Dividing by U^2/L, we get

$$1 \qquad 1 \qquad \frac{P}{\rho U^2} \qquad \frac{\nu}{UL}.$$

Definition (Reynolds number)**.** The *Reynolds number* is

$$Re = \frac{UL}{\nu},$$

which is a dimensionless number. This is a measure of the magnitude of the inertia to viscous terms.

Large Reynold number means almost no viscosity, while small Reynold's number means high viscosity. This number, rather than its individual constituents, determine the flow type.

Definition (Dynamic similarity)**.** Flows with the same geometry and equal Reynolds numbers are said to be dynamically similar.

Now let's look at two cases:

Low Re Number When this occurs, we ignore the left hand terms in the NS equations and write $0 = -\nabla p + \nu \nabla^2 \mathbf{u}$ with $\nabla \cdot \mathbf{u} = 0$ to keep incompressibility. This is the *Stokes' Equation*,

High Re Number When this occurs, the viscosity term is omitted and we have :

$$\rho \frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}\,t} = -\nabla p$$

With $|nabla \cdot \mathbf{u} = 0$. This is the *Euler Equations*, and to make this have solutions, we need to drop the no-slip condition, because we only have a first-order differential equation now.

3.4 Momentum Equation for Inviscid Incompressible Fluid

As we said above, if we ignore viscosity, we can treat this as the *Euler Equations.* Now we can derive this from momentum conservation in a region $\mathcal{D}$:

$$\begin{aligned} \text{Change in Momentum} \;&=\; \text{Momentum flow out of Boundary} \\ &+ \text{Surface Pressure Forces} \;+\; \text{Body Forces} \end{aligned}$$

In equation form, this is:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{D}} \rho \mathbf{u} \ \mathrm{d}V = - \int_{\partial \mathcal{D}} \rho \mathbf{u} (\mathbf{u} \cdot \mathbf{n}) \ \mathrm{d}S - \int_{\partial \mathcal{D}} p \mathbf{n} \ \mathrm{d}S + \int_{\mathcal{D}} \mathbf{f} \ \mathrm{d}V.$$

It is helpful to write this in suffix notation. In this case, the equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{D}} \rho u_i \ \mathrm{d}V = - \int_{\partial \mathcal{D}} \rho u_i u_j n_j \ \mathrm{d}S - \int_{\partial \mathcal{D}} -p n_i \ \mathrm{d}S + \int_{\mathcal{D}} f_i \ \mathrm{d}V.$$

Just as in the case of mass conservation, we can use the divergence theorem to write

$$\int_{\mathcal{D}} \left(\rho \frac{\partial u_i}{\partial t} + \rho \frac{\partial}{\partial x_j} (u_i u_j) \right) \ \mathrm{d}V = \int_{\mathcal{D}} \left(- \frac{\partial p}{\partial x_i} + f_i \right) \ \mathrm{d}V.$$

Since $\mathcal{D}$ is arbitrary, we must have

$$\rho \frac{\partial u_i}{\partial t} + \rho \frac{\partial u_i}{\partial x_j} + \rho u_i \frac{\partial u_j}{\partial x_j} = - \frac{\partial p}{\partial x_i} + f_i.$$

Therefore, we have:

$$\rho \frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}\,t} = -\nabla p + \mathbf{f}$$

For steady flow, if we look at our equation and take the partial time derivative of $\mathbf{u}$ as 0, then we have:

Proposition (Momentum integral for steady flow)**.**

$$\int_{\partial \mathcal{D}} (\rho \mathbf{u} (\mathbf{u} \cdot \mathbf{n}) + p \mathbf{n} + \chi \mathbf{n}) \ \mathrm{d}S = 0.$$

Now, noting that $\mathbf{u} \times (\nabla \times \mathbf{u}) = \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) - \mathbf{u} \cdot \nabla \mathbf{u}$, we can write the Euler equations as:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \nabla \left(\frac{1}{2}|\mathbf{u}|^2\right) - \rho \mathbf{u} \times (\nabla \times \mathbf{u}) = -\nabla p - \nabla \chi.$$

Where we just expanded the definition of a material derivative. Now we dot with $\mathbf{u}$ to get:

Proposition (Bernoulli's equation)**.**

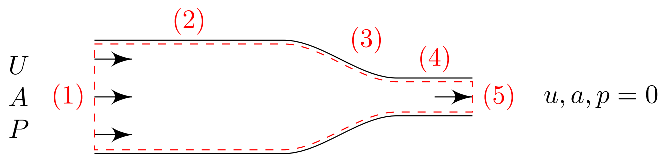
$$\frac{1}{2}\rho \frac{\partial |\mathbf{u}|^2}{\partial t} = -\mathbf{u} \cdot \nabla \left(\frac{1}{2}\rho|\mathbf{u}|^2 + p + \chi\right).$$

For a steady flow, we then have

$$H = \frac{1}{2}\rho|\mathbf{u}|^2 + p + \chi$$

is constant along streamlines. Now we would do an example involving the steady-flow equation and the Bernoulli equation:

Example (Force on a fire hose nozzle)**.** Suppose we have a fire hose nozzle like this:



We consider the steady-flow equation and integrate along the surface indicated above. We integrate each section separately. The end (1) contributes

$$\rho U(-U)A - PA.$$

On (2), everything vanishes. On (3), the first term vanishes since the velocity is parallel to the surface. Then we get a contribution of

$$0 + \int_{\text{nozzle}} p\mathbf{n} \cdot \hat{\mathbf{x}} \, \mathrm{d}S. \tag{3}$$

Similarly, everything in (4) vanishes. Finally, on (5), noting that $p = 0$, we get

$$\rho u^2 a.$$

By the steady flow equation, we know these all sum to zero. Hence, the force on the nozzle is just

$$F = \int_{\text{nozzle}} p\mathbf{n} \cdot \hat{\mathbf{x}} \, \mathrm{d}S = \rho A U^2 - \rho a u^2 + PA.$$

We can again apply Bernoulli along a streamline in the middle, which says

$$\frac{1}{2}\rho U^2 + P = \frac{1}{2}\rho u^2.$$

So we can get

$$F = \rho A U^2 - \rho a u^2 + \frac{1}{2}\rho A(u^2 - U^2) = \frac{1}{2}\rho \frac{A}{a^2} q^2 \left(1 - \frac{a}{A}\right)^2.$$

3.5 Linear Flow, or the Art of Approximation

Let's say we expand the flow $\mathbf{u}$ at some specific point in time. We use the Taylor expansion:

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x_0}) + (\mathbf{x} - \mathbf{x_0}) \cdot \nabla \mathbf{u} + \cdots$$

Wait. What is a gradient of a vector field? This is basically the Jacobian matrix. And we split this into symmetric and antisymmetric parts:

$$\nabla \mathbf{u} = \frac{\partial u_i}{\partial x_j} = E_{ij} + \Omega_{ij} = E + \Omega,$$

where

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

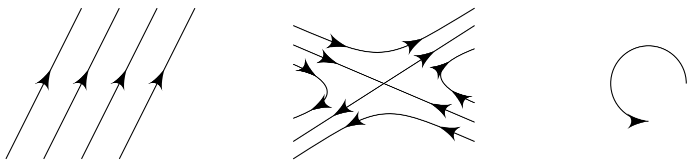
Remember vorticity? The term we defined long ago and then we never used it? Well turns out if we take $\mathbf{r} = \mathbf{x} - \mathbf{x_0}$, we have:

$$\boldsymbol{\omega} \times \mathbf{r} = (\nabla \times \mathbf{u}) \times \mathbf{r} = r_j \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = 2\Omega_{ij} r_j.$$

So we can write:

$$\mathbf{u} = \mathbf{u}_0 + E\mathbf{r} + \frac{1}{2}\boldsymbol{\omega} \times \mathbf{r}.$$

If we ignore quadratic terms and such. The first component is just uniform flow, the second component is the strain field (stress), and the last one is rotation:



uniform flow — pure strain — pure rotation

We would also note that for an incompressible fluid, we have $\nabla \cdot \mathbf{u} = 0$, so $E_{ii} = 0$ (sum).

3.6 Vorticity Equation

Well first we present the NS equations once again:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla|\mathbf{u}^2| - \mathbf{u} \times \boldsymbol{\omega}\right) = -\nabla p - \nabla \chi + \mu \nabla^2 \mathbf{u}$$

Noting that we used a vector calculus identity again. Then we take the curl of it, noting that the curl of a gradient just disappears:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} = \nu \nabla^2 \boldsymbol{\omega}.$$

We see another material derivative here, so we simplify it to reach the equation:

Proposition (Vorticity equation)**.**

$$\frac{\mathrm{D}\,\boldsymbol{\omega}}{\mathrm{D}\,t} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

4 Laplace's Equation, or Incompressible Inviscid Irrotational Flow

Ok. We make one more simplification, which is that there is no rotation in the flow. So that $\nabla \times \mathbf{u} = 0$. This, of course, also means that vorticity has to go. Sorry, its just too complicated. Now since we have $\nabla \times \mathbf{u}$ and $\nabla \cdot \mathbf{u}$ both 0, we can have a scalar potential ϕ with...:

$$\nabla^2 \phi = 0.$$

So the potential satisfies Laplace's equation. Surprise!

4.1 Three-dimensional Potential Flows

Note. It is *infinitely useful* to try to remember the Laplacian and the gradient in spherical polar coordinates. Examiners are not always nice so be prepared to write them down directly. They are:

$$\nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2}.$$

Note to people taking QM: If we take $r = 1$ and throw away the r partial derivative, we have $\mathbf{L}^2$, the angular momentum operator in Spherical Polars. So why not just remember it?

$$\mathbf{u} = \nabla \phi = \left(\frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \right).$$

Let's start with the boring stuff.

4.1.1 Only r Dependence

What if ϕ only depends on r? Then we have:

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) = 0 \qquad \Rightarrow \qquad \phi = -\frac{A}{r} + B$$

Yes, potentials are not unique, as we can just take a random constant somewhere. But just like voltage (EM), we can take $B = 0$ just to make lives easier:

$$\phi = -\frac{A}{r}.$$

What is the physical significance of the factor A? Consider the volume flux q across the surface of the sphere $r = a$. Then

$$q = \int_S \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S = \int_S u_r \, \mathrm{d}S = \int_S \frac{\partial \phi}{\partial r} \, \mathrm{d}S = \int_S \frac{A}{a^2} \, \mathrm{d}S = 4\pi A.$$

So we can write

$$\phi = -\frac{q}{4\pi r}.$$

When $q > 0$, this corresponds to a point source of fluid. When $q < 0$, this is a point sink of fluid.

4.1.2 r and θ Dependence

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0.$$

From IB Methods, we know that:

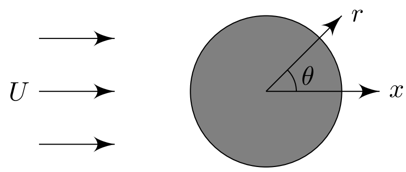
$$\phi = \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-n-1})P_n(\cos\theta).$$

We then have

$$\mathbf{u} = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, 0\right).$$

Now we would look at a classic example:

Example. We can look at uniform flow past a sphere of radius a.



We suppose the upstream flow is $\mathbf{u} = U\hat{\mathbf{x}}$. So

$$\phi = Ux = Ur\cos\theta.$$

So we need to solve

$$\begin{aligned} \nabla^2\phi &= 0 & r &> a\\ \phi &\to Ur\cos\theta & r &\to \infty\\ \frac{\partial\phi}{\partial r} &= 0 & r &= a. \end{aligned}$$

The last condition is there to ensure no fluid flows into the sphere, ie. $\mathbf{u}\cdot\mathbf{n} = 0$, for $\mathbf{n}$ the outward normal.

Since $P_1(\cos\theta) = \cos\theta$, and the P_n are orthogonal, our boundary conditions at infinity require ϕ to be of the form

$$\phi = \left(Ar + \frac{B}{r^2}\right)\cos\theta.$$

We now just apply the two boundary conditions. The condition that $\phi \to Ur\cos\theta$ tells us $A = U$, and the other condition tells us

$$A - \frac{2B}{a^3} = 0.$$

So we get

$$\phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta.$$

We can interpret $Ur\cos\theta$ as the uniform flow, and $U\frac{a^3}{2r^2}\cos\theta$ as the dipole response due to the sphere.

Note (EM People)**.** Yes. This is same as a conductor sphere in an electric field. In some way, the conductor is in a *fluid*.

We can compute the velocity to be

$$u_r = \frac{\partial \phi}{\partial r} = U\left(1 - \frac{a^3}{r^3}\right)\cos\theta$$

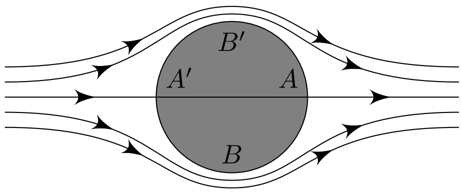
$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta.$$

We notice that $u_r = u_\theta = 0$ when $\phi = 0, \pi$ and $r = a$.

At the north and south poles, when $\theta = \pm\frac{\pi}{2}$, we get

$$u_r = 0, \quad u_\theta = \pm\frac{3U}{2}.$$

So the velocity is faster at the top than at the infinity boundary. This is why it is windier at the top of a hill than below.



We now apply Bernoulli's equation on a streamline to the surface (a, θ). Remember. Solid surfaces are *always* streamlines.

Comparing with what happens at infinity, we get

$$p_\infty + \frac{1}{2}\rho U^2 = p + \frac{1}{2}\rho U^2 \frac{9}{4}\sin^2\theta.$$

Thus, the pressure at the surface is

$$p = p_\infty + \frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right).$$

Note that the pressure is a function of $\sin^2\theta$. So the pressure at the back of the sphere is exactly the same as that at the front. Thus, if we integrate the pressure around the whole surface, we get 0. So the fluid exerts no net force on the sphere! This is d'Alemberts' paradox. This is why viscosity matters in Real Life^{TM}.

4.2 Two Dimensional Potential Flow

Now we downgrade a bit and look at the two-dimensional potential flow. In polar coordinates, we have:

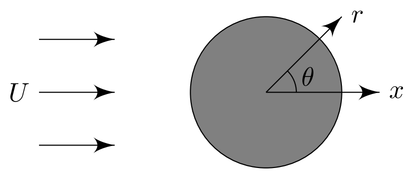
$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \quad \mathbf{u} = \nabla\phi = \left(\frac{\partial \phi}{\partial r}, \frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

The general solution is:

$$\phi = A\log r + B\theta + \sum_{n=1}^{\infty}(A_n r^n + B_n r^{-n})\begin{cases}\cos n\theta \\ \sin n\theta\end{cases}.$$

Now we look at the flow pass a cylinder:

Example (Uniform flow past a cylinder).



We need to solve

$$\begin{aligned} \nabla^2\phi &= 0 & r &> a \\ \phi &\to Ur\cos\theta & r &\to \infty \\ \frac{\partial \phi}{\partial r} &= 0 & r &= a. \end{aligned}$$

Similar to the 3D sphere, we have:

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta + \frac{K}{2\pi}\theta.$$

The last term allows for a net circulation K around the cylinder, to account for vorticity in the viscous boundary layer on the surface of the cylinder.

Note. Wait, aren't we in a section with absolutely no vorticity and stuff? Yes, that is still true. The only point with vorticity is the point where $r = 0$, and that has "infinite vorticity". This is how the term arises.

We have

$$\begin{aligned} u_r &= U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \\ u_\theta &= -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{K}{2\pi r}. \end{aligned}$$

We can find the streamfunction for this as

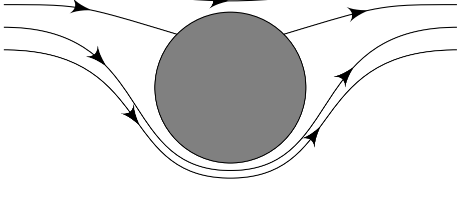
$$\psi = Ur\sin\theta\left(1 - \frac{a^2}{r^2}\right) - \frac{K}{2\pi}\log r.$$

If there is no circulation, ie. $K = 0$, then we get a flow similar to the flow around a sphere.

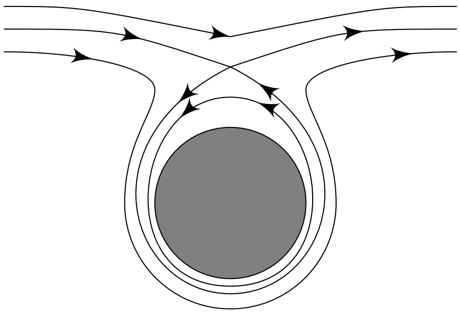
If $K \neq 0$ then we first look at the stagnation points. We get $u_r = 0$ if and only if $r = a$ or $\cos\theta = 0$. For $u_\theta = 0$, when $r = a$, we require

$$K = 4\pi a U \sin\theta.$$

So provided $|K| \leq 4\pi a U$, there is a solution to this problem, and we get stagnation points on the boundary.



For $|K| > 4\pi aU$, we do not get a stagnation point on the boundary. However, we still have the stagnation point where $\cos\theta = 0$, ie. $\theta = \pm\frac{\pi}{2}$. Looking at the equation for $u_\theta = 0$, only $\theta = \frac{\pi}{2}$ works.



Let's now look at the effect on the sphere. For steady potential flow, Bernoulli works (ie. H is constant) everywhere, not just along each streamline. So we can calculate the pressure on the surface. Let p be the pressure on the surface. Then we get

$$p_\infty + \frac{1}{2}\rho U^2 = p + \frac{1}{2}\rho\left(\frac{K}{2\pi a} - 2U\sin\theta\right)^2.$$

So we find

$$p = p_\infty + \frac{1}{2}\rho U^2 - \frac{\rho K^2}{8\pi^2 a^2} + \frac{\rho K U\sin\theta}{\pi a} - 2\rho U^2 \sin^2\theta.$$

We see the pressure is symmetrical. So there is no force in the x direction.

However, we get a transverse force (per unit length) in the y-direction. We have

$$F_y = -\int_0^{2\pi} p\sin\theta(a\ \mathrm{d}\theta) = -\int_0^{2\pi} \frac{\rho KU}{\pi a}\sin^2\theta a\ \mathrm{d}\theta = -\rho UK.$$

where we have dropped all the odd terms. So there is a sideways force in the direction perpendicular to the flow, and is directly proportional to the circulation of the system.

In general, the *magnus force* (lift force) resulting from interaction between the flow $\mathbf{U}$ and the vortex $\mathbf{K}$ is

$$\mathbf{F} = \rho\mathbf{U}\times\mathbf{K}.$$

4.3 Time-dependent Potential Flows

We consider the time-dependent Euler equation:

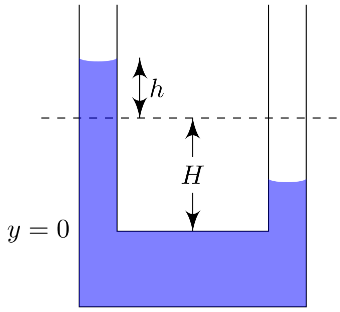
$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) - \mathbf{u}\times\boldsymbol{\omega}\right) = -\nabla p - \nabla\chi.$$

And since we have a potential flow, $\boldsymbol{\omega} = 0$. Then, we have:

$$\nabla\left(\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho|\mathbf{u}|^2 + p + \chi\right) = 0 \quad\Rightarrow\quad \rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho|\nabla\phi|^2 + p + \chi = f(t)$$

So we can equate different points in space at the same time. We would look at an example:

Example (Oscillations in a manometer)**.** A manometer is a U-shaped tube.



We used some magic to set it up such that the water level in the left tube is h above the equilibrium position H. Then when we release the system, the water levels on both side would oscillate.

We are going to assume the reservoir at the bottom is large, so velocities are negligible. So ϕ is constant in the reservoir, say $\phi = 0$. We want to figure out the velocity on the left. This only moves vertically. So we have

$$\phi = uy = \dot{h}y.$$

So we have

$$\frac{\partial \phi}{\partial t} = \ddot{h}y.$$

On the right hand side, we just have

$$\phi = -uy = -\dot{g}y, \quad \frac{\partial \phi}{\partial t} = -\ddot{h}y.$$

We now apply the equation from one tube to the other — we get

$$\begin{aligned} \rho\ddot{h}(H + h) + \frac{1}{2}\rho\dot{h}^2 + p_{\mathrm{atm}} + g\rho(H + h) &= f(t) \\ &= -\rho\ddot{h}(H - h) + \frac{1}{2}\rho\dot{h}^2 + p_{\mathrm{atm}} + g\rho(H - h). \end{aligned}$$

Quite a lot of these terms cancel, and we are left with

$$2\rho H\ddot{h} + 2g\rho h = 0.$$

Simplifying terms, we get

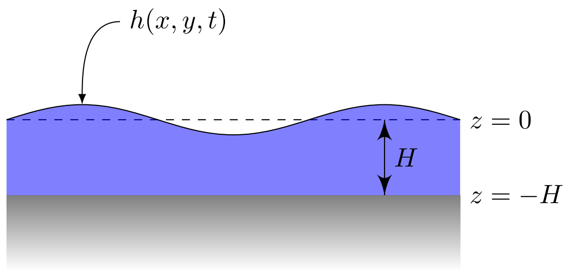
$$\ddot{h} + \frac{g}{H}h = 0.$$

So this is simple harmonic motion with the frequency $\sqrt{\frac{g}{H}}$.

5 Water Waves

5.1 Equation and boundary conditions

We now try to solve for the actual solution.



We assume the fluid is inviscid, and the motion starts from rest. Thus the vorticity $\nabla \times \mathbf{u}$ is initially zero, and hence always zero. Together with the incompressibility condition $\nabla \cdot \mathbf{u} = 0$, we end up with Laplace's equation

$$\nabla^2 \phi = 0.$$

We have some *kinematic boundary conditions*. First of all, there can be no flow through the bottom. So we have

$$u_z = \frac{\partial \phi}{\partial z} = 0$$

when $z = -H$. At the free surface, we have

$$u_z = \frac{\partial \phi}{\partial z} = \frac{\mathrm{D}\, h}{\mathrm{D}\, t} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}$$

when $z = h$.

Note. The equation above is not magical. If we *assume* that particles on the surface stays on the surface, then the particle movement must be same as the movement of the wave. This is like the no-slip condition.

We then have the dynamic boundary condition that the pressure at the surface is the atmospheric pressure, ie. at $z = h$, we have

$$p = p_0 = \text{constant}.$$

We need to relate this to the flow. So we apply the time-dependent Bernoulli equation

$$\rho\frac{\partial \phi}{\partial t} + \frac{1}{2}\rho|\nabla \phi|^2 + g\rho h + p_0 = f(t) \text{ on } z = h.$$

Now we assume

$$h \ll H \quad \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \ll 1$$

We then ignore quadratic terms in small quantities. For example, since the waves are small, the velocities u and v also are. So we ignore $u\frac{\partial h}{\partial x}$ and $v\frac{\partial h}{\partial y}$. Similarly, we ignore the whole of $|\nabla \phi|^2$ in Bernoulli's equations since it is small.

Next, we use Taylor series to write

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=h} = \left.\frac{\partial \phi}{\partial z}\right|_{z=0} + h\left.\frac{\partial^2 \phi}{\partial z^2}\right|_{z=0} + \cdots .$$

Again, we ignore all quadratic terms. So we just approximate

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=h} = \left.\frac{\partial \phi}{\partial z}\right|_{z=0}.$$

We are then left with linear water waves. The equations are then

$$\begin{aligned}
\nabla^2\phi &= 0 & -H < z \leq 0\\
\frac{\partial \phi}{\partial z} &= 0 & z = -H\\
\frac{\partial \phi}{\partial z} &= \frac{\partial h}{\partial t} & z = 0\\
\frac{\partial \phi}{\partial t} + gh &= f(t) & z = h.
\end{aligned}$$

Note that the last equation is just Bernoulli equations, after removing the small terms and throwing our constants and factors in to the function f.

We now have a nice, straightforward problem. We have a linear equation with linear boundary conditions, which we can solve.

Also note that $f(t) = 0$ in many cases. Why? We have $\frac{\partial \phi}{\partial t}$ proportional to h for simple waves, and gh is proportional to h, which is a function os space and time. The function on the right is not proportional to h, so it must be 0.

5.2 Two-dimensional waves (straight crested waves)

We are going further simplify the situation by considering the case where the wave does not depend on y. We consider a simple wave form

$$h = h_0 e^{i(kx - \omega t)}.$$

Using the boundary condition at $z = 0$, we know we must have a solution of the form

$$\phi = \hat{\phi}(z) e^{i(kx - \omega t)}.$$

Putting this into Laplace's equation, we have

$$-k^2 \hat{\phi} + \hat{\phi}'' = 0.$$

We notice that the solutions are then of the form

$$\hat{\phi} = \phi_0 \cosh k(z + H),$$

where the constants are chosen so that $\frac{\partial \phi}{\partial z} = 0$ at $z = -H$.

We now have three unknowns, namely h_0, ϕ_0 and ω (we assume k is given, and we want to find waves of this wave number). We use the boundary condition

$$\frac{\partial \phi}{\partial z} = \frac{\partial h}{\partial t} \text{ at } z = 0.$$

We then get

$$k\phi_0 \sinh kH = -i\omega h_0.$$

We put in Bernoulli's equation to get

$$-i\omega \hat{\phi}(z) e^{i(kx - \omega t)} + g h_0 e^{i(kx - \omega t)} = f(t).$$

For this not to depend on x, we must have

$$-i\omega\phi_0 \cosh kH + g h_0 = 0.$$

The trivial solution is of course $h_0 = \phi_0 = 0$. Otherwise, we can solve to get

$$\omega^2 = gk \tanh kH.$$

This is the *dispersion relation*, and relates the frequency to the wavelengths of the wave.

We can use the dispersion relation to find the speed of the wave. This is just

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k} \tanh kH}.$$

We can now look at the limits we have previously obtained with large and small H.

In deep water (or short waves), we have $kH \gg 1$. We know that as $kH \to \infty$, we get $\tanh kH \to 1$. So we get

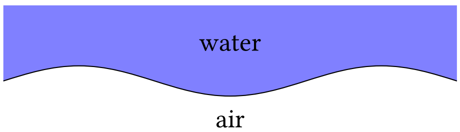
$$c = \sqrt{\frac{g}{k}}.$$

In shallow water, we have $kH \ll 1$. In the limit $kH \to 0$, we get $\tanh kH \to kH$. Then we get

$$c = \sqrt{gH}.$$

5.3 Rayleigh-Taylor instability

Note that it is possible to turn the problem upside down, and imagine we have water over air:



This is really the same scenario as before, but with gravity pull upwards. So exactly the same equations hold, but we replace $-g$ with g. We then look at the limits.

$$\omega^2 = -gk \quad \Rightarrow \quad \omega = \pm i\sqrt{gk}.$$

Thus we get

$$h \propto Ae^{\sqrt{gk}t} + Be^{-\sqrt{gk}t}.$$

We thus have an exponentially growing solution. So the system is unstable, and water will fall down. This very interesting fact is known as *Rayleigh-Taylor instability*.

6 Fluid Dynamics. Rotating.

The Lagrangian (particle) acceleration in a rotating frame of reference is given by

$$\frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}\,t} + 2\boldsymbol{\Omega} \times \mathbf{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}),$$

as you might recall from IA Dynamics and Relativity. So we have the equation of motion

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u}\right) = -\nabla p - \rho\boldsymbol{\omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) + \mathbf{g}\rho.$$

Now for earth, we incorporate the $\Omega \times (\Omega \times \mathbf{x})$ into the gravity, because it is always pointing radially outwards, and can be written as a scalar potential. Now we get rid of the non linear $\mathbf{u} \cdot \nabla \mathbf{u}$ by saying that this term is much smaller than $2|\mathbf{\Omega} \times \mathbf{u}|$ for not very fast objects. Then we have:

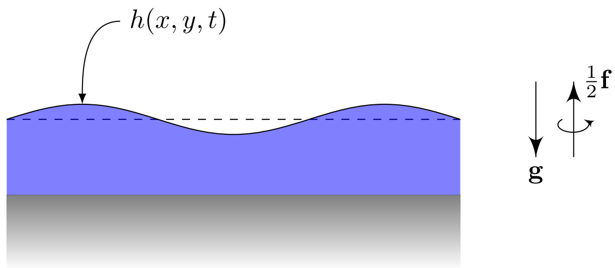
Proposition (Euler's equation in a rotating frame)**.**

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{f} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g}.$$

Where we have written $2\mathbf{\Omega} = \mathbf{f}$ as the *coriolis parameter*, or the planetary vorticity.

6.1 Shallow Water Equations

Now suppose we have a shallow layer of depth $z = h(x, y)$ with $p = p_0$ on $z = h$.



We consider motions with horizontal scales L much greater than vertical scales H.

We use the fact that the fluid is incompressible, ie. $\nabla \cdot \mathbf{u} = 0$. Writing $\mathbf{u} = (u, v, w)$, we get

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}.$$

The scales of the terms are W/H, U/L and V/L respectively. Since $H \ll L$, we know $W \ll U, V$, ie. most of the movement is horizontal, which makes sense, since there isn't much vertical space to move around.

We consider only horizontal velocities, and write

$$\mathbf{u} = (u, v, 0),$$

and

$$\mathbf{f} = (0, 0, f).$$

Basically we only care about the portion of coriolis force that causes an effect horizontally, so f is really $f \sin\theta$, where θ is the latitude. The vertical part? We incorporate that into gravity. Then from Euler's equations, we get

$$\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x}, \\
\frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y}, \\
0 &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g.
\end{aligned}$$

From the last equation, plus the boundary conditions, we know

$$p = p_0 = g\rho(h - z).$$

This is just the hydrostatic balance. We now put this expression into the horizontal components to get

$$\begin{aligned} \frac{\partial u}{\partial t} - fv &= -g\frac{\partial h}{\partial x}, \\ \frac{\partial v}{\partial t} + fu &= -g\frac{\partial h}{\partial y}. \end{aligned}$$

6.2 Geostrophic balance

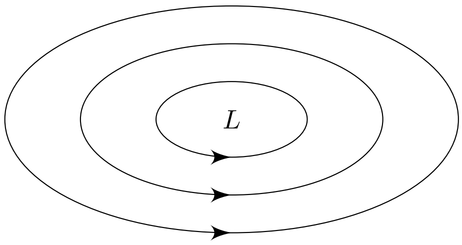
When we have steady flow, the time derivatives vanish. So we get

$$\begin{aligned} u &= \frac{\partial}{\partial y}\left(-\frac{gh}{f}\right) = \frac{\partial}{\partial y}\left(-\frac{p}{\rho f}\right), \\ v &= -\frac{\partial}{\partial x}\left(-\frac{gh}{f}\right) = -\frac{\partial}{\partial x}\left(-\frac{p}{\rho f}\right). \end{aligned}$$

Hopefully, this reminds us of streamfunctions. The streamlines are places where h is constant, ie. the surface is of constant height, ie. the pressure is constant.

Definition (Shallow water streamfunction)**.** The quantity $\psi = -\frac{gh}{f}$ is the *shallow water streamfunction.*

In general, near a low pressure zone, there is a pressure gradient pushing the flow towards to the low pressure area. Since flow moves in circles around the low pressure zone, there is a Coriolis force that balances this force. This is the geostrophic balance.



This is a cyclone. Note that this picture is only valid in the Northern hemisphere. If we are on the other side of the Earth, cyclones go the other way round.

We now look at conservation of mass.

We consider a horizontal surface $\mathcal{D}$ in the water. Then we can compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{D}} \rho h \;\mathrm{d}V = -\int_{\partial\mathcal{D}} h\rho\mathbf{u}_H \cdot \mathbf{n}\;\mathrm{d}S,$$

where $\mathbf{u}_H$ is the horizontal velocity. Applying the divergence theorem, we get

$$\int_{\mathcal{D}} \frac{\partial}{\partial t}(\rho h)\;\mathrm{d}V = -\int_{\mathcal{D}} \nabla_H \cdot (\rho h \mathbf{u}_H)\;\mathrm{d}V,$$

where

$$\nabla_H = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right).$$

Since this was an arbitrary surface, we can take the integral away, and we have the continuity equation

$$\frac{\partial h}{\partial t} + \nabla_H \cdot (\mathbf{u}_h h) = 0.$$

So if there is water flowing into a point (ie. a vertical line), then the height of the surface falls, and vice versa.

We can write this out in full. In Cartesian coordinates:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(uh) = 0.$$

To simplify the situation, we suppose we have small oscillations, so we have $h = h_0 + \eta(x, y, t)$, where $\eta \ll h_0$, and write

$$\mathbf{u} = (u(x, y), v(x, y)).$$

Then we can rewrite our equations of motion as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{f} \times \mathbf{u} = -g\nabla\eta \qquad (*)$$

and, ignoring terms like $u\eta, v\eta$, the continuity equation gives

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} + h_0 \nabla \cdot \mathbf{u} = 0. \qquad (\dagger)$$

Taking the curl of the $(*)$, we get

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} + \mathbf{f}\nabla \cdot \mathbf{u} = 0,$$

where

$$\boldsymbol{\zeta} = \nabla \times \mathbf{u}.$$

Note that even though we wrote this as a vector equation, really only the z-component is non-zero. So we can also view $\boldsymbol{\zeta}$ and $\mathbf{f}$ as scalars, and get a scalar equation.

We can express $\nabla \cdot \mathbf{u}$ in terms of η using $(\dagger)$. So we get

$$\frac{\partial}{\partial t}\left(\boldsymbol{\zeta} - \frac{\eta}{h_0}\mathbf{f}\right) = \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t} = 0,$$

where

Definition (Potential vorticity)**.** The *potential vorticity* is

$$\mathbf{Q} = \boldsymbol{\zeta} - \frac{\eta}{h_0}\mathbf{f},$$

and this is conserved.

Hence given any initial condition, we can compute $\mathbf{Q}(x, y, 0) = \mathbf{Q}_0$. Then we have

$$\mathbf{Q}(x, y, t) = \mathbf{Q}_0$$

for all time.

How can we make use of this? We start by taking the divergence of $(*)$ above to get

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{u}) - \mathbf{f} \cdot \nabla \times \mathbf{u} = -g\nabla^2\eta,$$

and use (†) to substitute

$$\nabla \cdot \mathbf{u} = -\frac{1}{h_0}\frac{\partial \eta}{\partial t}.$$

We then get

$$-\frac{1}{h_0}\frac{\partial^2 \eta}{\partial t^2} - \mathbf{f} \cdot \boldsymbol{\zeta} = -g\nabla^2 \eta.$$

We now use the conservation of potential vorticity, namely

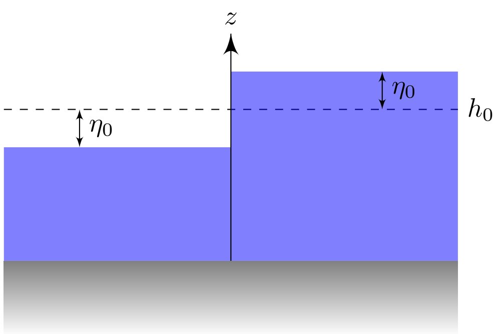
$$\boldsymbol{\zeta} = \mathbf{Q}_0 + \frac{\eta}{h_0}\mathbf{f},$$

to rewrite this as

$$\frac{\partial^2 \eta}{\partial t^2} - gh_0 \nabla^2 \eta + \mathbf{f} \cdot \mathbf{f}\eta = -h_0 \mathbf{f} \cdot \mathbf{Q}_0.$$

Note that the right hand side is just a constant (in time). So we have a nice differential equation we can solve.

Example. Suppose we have fluid with mean depth h_0, and we start with the following scenario:



Due to the differences in height, we have higher pressure on the right and lower pressure on the left.

If there is no rotation, then the final state is a flat surface with no flow. However, this cannot be the case if there is rotation, since this violates the conservation of $\mathbf{Q}$. So what happens if there is rotation?

At the beginning, there is no movement. So we have $\boldsymbol{\zeta}(t = 0) = 0$. Thus we have

$$Q_0 = \begin{cases} -\frac{\eta_0}{h_0} f & x > 0 \\ \frac{\eta_0}{h_0} f & x < 0 \end{cases}.$$

We seek the final steady state such that

$$\frac{\partial \eta}{\partial t} = 0.$$

We further assume that the final solution is independent of y, so

$$\frac{\partial \eta}{\partial y} = 0.$$

So $\eta = \eta(x)$ is just a function of x. Our equation then says

$$\frac{\partial^2 \eta}{\partial x^2} - \frac{f^2}{gh_0}\eta = \frac{f}{g}Q_0 = \mp \frac{f^2}{gh_0}\eta_0.$$

It is convenient to define a new variable

$$R = \frac{\sqrt{gh_0}}{f},$$

which is a length scale. We know $\sqrt{gh_0}$ is the fastest possible wave speed because of the asymptotic solutions we derived earlier, and thus R is how far a wave can travel in one rotation period. We rewrite our equation as

$$\frac{\mathrm{d}^2 \eta}{\mathrm{d}x^2} - \frac{1}{R^2}\eta = \mp \frac{1}{R^2}\eta_0.$$

Definition (Rossby radius of deformation)**.** The length scale

$$R = \frac{\sqrt{gh_0}}{f}$$

is the *Rossby radius of deformation.*

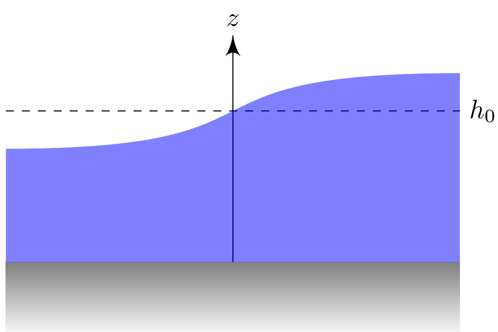
This is the fundamental length scale to use in rotating systems when gravity is involved as well.

We now impose our boundary conditions. We require $\eta \to \pm\eta_0$ as $x \to \pm\infty$. We also require η and $\frac{\mathrm{d}\eta}{\mathrm{d}x}$ to be continuous at $x = 0$.

The solution is

$$\eta = \eta_0 \begin{cases} 1 - e^{-x/R} & x > 0 \\ -(1 - e^{x/R}) & x < 0 \end{cases}.$$

We can see that this looks quite different form the non-rotating case. It looks like this:



The horizontal length scale involved is $2R$.

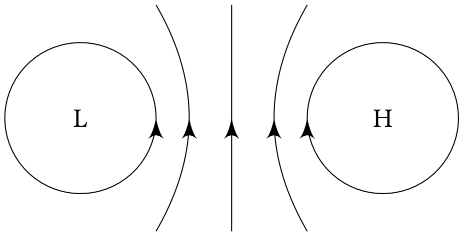
We now look at the velocities. Using the steady flow equations, we have

$$u = -\frac{g}{f}\frac{\partial \eta}{\partial y} = 0$$
$$v = \frac{g}{f}\frac{\partial \eta}{\partial x} = \eta_0 \sqrt{\frac{g}{h_0}} e^{-|x|/R}.$$

So there is still flow in this system, and is a flow in the y direction into the paper. This flow gives Coriolis force to the right, and hence balances the pressure gradient of the system.

The final state is not one of rest, but one with motion in which the Coriolis force balances the pressure gradient. This is geostrophic flow.

Going back to our pressure maps, if we have high and low pressure systems, we can have flows that look like this:



Then the Coriolis force will balance the pressure gradients.