Electromagnetism Review Sheet

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Electromagnetism and Relativity
Review of Special Relativity; tensors and index notation. Lorentz force law. Electromagnetic tensor. Lorentz transformations of electric and magnetic fields. Currents and the conservation of charge. Maxwell equations in relativistic and non-relativistic forms. [5]

Electrostatics
Gauss's law. Application to spherically symmetric and cylindrically symmetric charge distributions. Point, line and surface charges. Electrostatic potentials; general charge distributions, dipoles. Electrostatic energy. Conductors. [3]

Magnetostatics
Magnetic fields due to steady currents. Ampre's law. Simple examples. Vector potentials and the Biot-Savart law for general current distributions. Magnetic dipoles. Lorentz force on current distributions and force between current-carrying wires. Ohm's law. [3]

Electrodynamics
Faraday's law of induction for fixed and moving circuits. Electromagnetic energy and Poynting vector. 4-vector potential, gauge transformations. Plane electromagnetic waves in vacuum, polarization. [5]

Contents

1 Introduction

Definition (Charge and Current)**.** We define the *charge density*, $\rho(\mathbf{x}, t)$ as the charge per unit volume, and the *current density* $\mathbf{J} = \rho\mathbf{v}$, where $\mathbf{v}$ is the velocity. Then we define the charge Q and current I as followed:

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} \qquad Q = \int_V \rho(\mathbf{x}, t)\ \mathrm{d}V$$

Now we would introduce some fundmanetal laws of nature:

Law (Continuity equation)**.**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = -\int_S \mathbf{J} \cdot \mathrm{d}S.$$

Law (Lorentz force law)**.**

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

while the second aspect is governed by *Maxwell's equations.*

Law (Maxwell's Equations)**.**

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

where $\epsilon_0 = 8.85 \times 10^{-12} m^{-3} kg^{-1} s^2 C^2$ is the electric constant and $\mu_0 = 4\pi * 10^{-6} m \cdot kg \cdot C^{-2}$ is the magnetic constant.

2 Electrostatics

This is basically the maxwell equations taking $\mathbf{B} = 0$ and $\mathbf{J} = 0$. The first one we would tackle is $\nabla \cdot \mathbb{E} = \frac{\rho}{\epsilon_0}$:

2.1 Gauss' Law

If we integrate the maxwell equation over a surface V, we obtain Gauss' law using the divergence theorem:

Law (Gauss' law)**.**

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\varepsilon_0},$$

where Q is the total charge inside V.

Now we define the term on the left:

Definition (Flux through surface)**.** The *flux* of $\mathbf{E}$ through the surface S is defined to be

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}.$$

The only important thing in calculating the electric field for these surfaces is to use the right symmetry and pick the right surface. We would list the results of some common examples:

Sphere Using spherical symmetry and picking a radius r from a ρ uniform charge R-radius sphere, we have:

$$\mathbf{E} = \begin{cases} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} & r > R \\ \frac{Qr}{4\pi\varepsilon_0 R^3}\hat{\mathbf{r}} & r \leq R \end{cases}$$

Infinite line Using circular symmetry, where r is now the r in cylindrical coordinates, we take a cylinder around the line and reach that:

$$\mathbf{E}(r) = \frac{\eta}{2\pi\varepsilon_0 r}\hat{\mathbf{r}}.$$

Infinite Plane Now we have vertical symmetry, and we can take a box that is symmetric about the plane to get $E(z)$:

$$E(z) = \frac{\sigma}{2\varepsilon_0}$$

Where it point upwards above the plane and downwards below.

2.2 Electrostatic Potential

From $\nabla \times \mathbf{E} = 0$, we know that $\mathbf{E} = -\nabla\phi$ for some *electrostatic potential* ϕ. Substituting into $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$, we have:

$$\nabla^2\phi = \frac{\rho}{\varepsilon_0}.$$

We now list some examples:

Point Charge This gives $\rho = Q\delta^3(\mathbf{r})$, where δ^3 is the 3D delta function. Now we can solve this to give $\phi = \frac{Q}{4\pi\epsilon_0 r}$ as from 1A Vector Calculus we know the solution is some multiple of $\frac{1}{r}$. Differentiating it gives Coulomb's law.

Dipole We have:

$$\phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} - \frac{Q}{|\mathbf{r}+\mathbf{d}|}\right).$$

From superposition of point charges. We taylor expand $|\mathbf{R} + \mathbf{d}$ to be:

$$\begin{aligned} \frac{1}{|\mathbf{r}+\mathbf{d}|} &= \frac{1}{r} - \mathbf{d}\cdot\nabla\left(\frac{1}{r}\right) + \frac{1}{2}(\mathbf{d}\cdot\nabla)^2\left(\frac{1}{r}\right) + \cdots \\ &= \frac{1}{r} - \frac{\mathbf{d}\cdot\mathbf{r}}{r^3} - \frac{1}{2}\left(\frac{\mathbf{d}\cdot\mathbf{d}}{r^3} - \frac{3(\mathbf{d}\cdot\mathbf{r})^2}{r^5}\right) + \cdots. \end{aligned}$$

Then this gives:

$$\phi = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{r} + \frac{\mathbf{d}\cdot\mathbf{r}}{r^3} + \cdots\right) \sim \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{d}\cdot\mathbf{r}}{r^3}.$$

If we define the *electric dipole moment* to be $\mathbf{p} = Q\mathbf{d}$, then we have:

$$\mathbf{E} = -\nabla\phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3}\right).$$

General To find ϕ for a general charge distribution ρ, we use the Green's function for the Laplacian. The Green's function is defined to be the solution to

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta^3(\mathbf{r} - \mathbf{r}'), \quad \text{or} \quad G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}'|}.$$

We assume all charge is contained in some compact region V. Then

$$\begin{aligned} \phi(\mathbf{r}) &= -\frac{1}{\varepsilon_0} \int_V \rho(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') \, \mathrm{d}^3\mathbf{r}' \\ &= \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}^3\mathbf{r}' \end{aligned}$$

Then

$$\begin{aligned} \mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) &= -\frac{1}{4\pi\varepsilon_0} \int_V \rho(\mathbf{r}') \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \, \mathrm{d}^3\mathbf{r}' \\ &= \frac{1}{4\pi\varepsilon_0} \int_V \rho(\mathbf{r}') \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, \mathrm{d}^3\mathbf{r}' \end{aligned}$$

Now, to make it simple, we can ask what ϕ and $\mathbf{E}$ look like very far from V, ie. $|\mathbf{r}| \gg |\mathbf{r}'|$ using taylor expansion:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \cdots.$$

Then we get

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0} \int_V \rho(\mathbf{r}') \left(\frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \cdots \right) \, \mathrm{d}^3\mathbf{r}' \\ &= \frac{1}{4\pi\varepsilon_0} \left(\frac{Q}{r} + \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} + \cdots \right), \end{aligned}$$

where

$$Q = \int_V \rho(\mathbf{r}') \, \mathrm{d}V' \quad \mathbf{p} = \int_V \mathbf{r}' \rho(\mathbf{r}') \, \mathrm{d}V' \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{\|\mathbf{r}\|}.$$

So if we have a huge lump of charge, we can consider it to be a point charge Q, plus some dipole correction terms when we are *far* away from the charge.

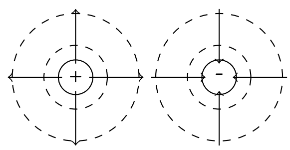
2.3 Field Lines and Equipotentials

Let's start from some definitions:

Definition (Field line)**.** A *field line* is a continuous line tangent to the electric field $\mathbf{E}$. The density of lines is proportional to $|\mathbf{E}|$.

Definition (Equipotentials)**.** *Equipotentials* are surfaces of constant ϕ. Because $\mathbf{E} = -\nabla\phi$, they are always perpendicular to field lines.

if we indicate field lines using straight lines and equipotentials using dashed ones, we have:



2.4 Electrostatic Energy

If we want to calculate the amount of energy stored (potential energy), then it is the sum of work done to bring the charges from infinity. Since we can bring each charge one by one, and between two point charges i and j, the work done is:

$$\frac{q_i q_j}{4\pi\epsilon_0} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j}$$

The total potential energy is:

$$U = \frac{1}{4\pi\varepsilon_0} \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} = \frac{1}{4\pi\varepsilon_0} \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}.$$

Since the potential at each point is due to all other points:

$$\phi(\mathbf{r}_i) = \frac{1}{4\pi\varepsilon_0} \sum_{j \neq i} \frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|}.$$

We can write

$$U = \frac{1}{2} \sum_{i=1}^{N} q_i \phi(\mathbf{r}_i).$$

Thus, for continuous charge distributions, we have:

$$\begin{aligned} U = \frac{1}{2} \int \rho(\mathbf{r}) \phi(\mathbf{r}) \, \mathrm{d}^3\mathbf{r} &= \frac{\varepsilon_0}{2} \int (\nabla \cdot \mathbf{E}) \phi \, \mathrm{d}^3\mathbf{r} \\ &= \frac{\varepsilon_0}{2} \int [\nabla \cdot (\mathbf{E}\phi) - \mathbf{E} \cdot \nabla\phi] \, \mathrm{d}^3\mathbf{r} \\ &= \frac{\varepsilon_0}{2} \int \mathbf{E} \cdot \mathbf{E} \, \mathrm{d}^3\mathbf{r}. \end{aligned}$$

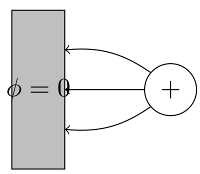
$\int \nabla(E\phi)$ vanishes as we assume $\phi \to 0$ as we go to infinity.

2.5 Conductors, or the Image Charge Method

Definition (Conductor)**.** A *conductor* is a region of space which contains lots of charges that are free to move.

In electrostatic situations, we must have $\mathbf{E} = 0$ inside a conductor. Why? Since if there is an electric field, charges move around to balance it. And after charges has balanced the field, there is no field! Therefore all charges in a conductor must be on the surface. To consider what happens when a charge comes near to a surface, we consider an example:

Example. Suppose we have a conductor that fills all space $x < 0$. We ground it such that $\phi = 0$ throughout the conductor. Then we place a charge q at $x = d > 0$.



We look for a potential that has a source at $x = d$ and satisfies $\phi = 0$ for $x < 0$. Since the solution to the Poisson equation is unique, we can use the method of images, which is to pretend that we don't have a conductor. Instead, we have a charge $-q$ and $x = -d$. Then by symmetry we will get $\phi = 0$ when $x = 0$. The potential of this pair is

$$\phi = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{q}{\sqrt{(x+d)^2+y^2+z^2}}\right].$$

To get the solution we want, we "steal" part of this potential and declare our potential to be

$$\phi = \begin{cases} \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{q}{\sqrt{(x+d)^2+y^2+z^2}}\right] & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Using this solution, we can immediately see that it satisfies Poisson's equations both outside and inside the conductor.

Now we calculate the surface charge induced. We can calculate the electric field near the surface, and use the relation $\sigma = \mathbf{E}_{\text{outside}} \cdot \hat{\mathbf{n}}$. To find σ, we only need the component of $\mathbf{E}$ in the x direction:

$$\mathbf{E}_x = -\frac{\partial\phi}{\partial x} = \frac{q}{4\pi\varepsilon_0}\left(\frac{x-d}{|\mathbf{r}-\mathbf{d}|^{3/2}} - \frac{x+d}{|\mathbf{r}+\mathbf{d}|^{3/2}}\right)$$

for $x > 0$. Then induced surface charge density is given by $\mathbf{E}_x$ at $x = 0$:

$$\sigma = E_x\varepsilon_0 = -\frac{q}{2\pi}\frac{d}{(d^2+y^2+z^2)^{3/2}}.$$

The total surface charge is then given by

$$\int \sigma \, \mathrm{d}y \, \mathrm{d}z = -q.$$

Which is expected.

Note. Here are some handy tips for handling other situations:

Sphere Image Charge: For a grounded sphere with radius R and a charge at $(d, 0, 0)$, we should place the image charge at $x = \frac{R^2}{d}$.

Conductor in Field: For a spherical conductor in a constant electric field, we orient the electric field in the z-axis to give $\phi_0 = -E_0 r\cos\theta$ and add a dipole term to the potential, giving:

$$\phi = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$$

Note. For people learning fluids, if this looks like the solution of a sphere passing through a fluid, because this is! This is solving the Laplace equation with $\phi = 0$ on the spherical boundary.

3 Magnetostatics

Now similarly, we look at $\mathbf{E} = 0$. Then we are left with:

$$\nabla \times \mathbf{B} = \mu\mathbf{J} \qquad \nabla \cdot \mathbf{B} = 0$$

3.1 Ampere's Law

We use Stokes' Law to integrate the first equation to get:

Law (Ampere's law).

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{r} = \mu_0 I,$$

where I is the current through the surface.

Similarly, we have the classic examples:

Long Straight Wire We integrate over a disc to find:

$$\mathbf{B}(r) = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\varphi}}.$$

Infinite surface Now if we have constant x direction current on an infinite plane with surface current density $\mathbf{k}$, then we must have y direction magnetic field. So we choose a loop through the surface to find:

$$B(z) = \frac{\mu_0 k}{2} \quad \text{for } z > 0.$$

Similarly, the direction reverses for above/below the plane.

Solenoid Imagine wrapping N wires per unit length with each wire having current I. Then we integrate over a loop that passes the cylinder, as the magnetic field must be in the z direction inside the cylinder:

$$\mathbf{b} = \mu_0 I N \mathbf{z}.$$

inside the cylinder. Outside the cylinder, $\mathbf{B}$ is constant, so it is 0.

3.2 Vector Potential

Definition (Vector potential)**.** If $\mathbf{B} = \nabla \times \mathbf{A}$, then $\mathbf{A}$ is *a vector potential.*

This is *NOT* unique! If A is a vector potential, then $\mathbf{A}' = \mathbf{A} + \nabla\chi$ is another one. But we do have some canonical choice:

Definition ((Coulomb) gauge)**.** Each choice of $\mathbf{A}$ is called a *gauge.* An $\mathbf{A}$ such that $\nabla \cdot \mathbf{A} = 0$ is called a *Coulomb gauge.*

Proposition. We can always pick χ such that $\nabla \cdot \mathbf{A}' = 0$.

Proof. Suppose that $\mathbf{B} = \nabla \times \mathbf{A}$ with $\nabla \cdot \mathbf{A} = \psi(\mathbf{x})$. Then for any $\mathbf{A}' = \mathbf{A} + \nabla\chi$, we have

$$\nabla \cdot \mathbf{A}' = \nabla \mathbf{A} + \nabla^2\chi = \psi + \nabla^2\chi.$$

So we need a χ such that $\nabla^2\chi = -\psi$. This is the Poisson equation which we know that there is always a solution by, say, the Green's function. Hence we can find a χ that works. □

Now the other maxwell equation says:

$$\nabla \times \mathbf{B} = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = -\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}. \tag{$*$}$$

Which we can solve this using Green functions to get

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}V',$$

Note this is integrating over $\mathbf{r}'$, not $\mathbf{r}$. Fortunately, this does satisfy the Coulomb gauge:

$$\begin{aligned} \nabla \cdot \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \cdot \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \, \mathrm{d}V' \\ &= -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \cdot \nabla' \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \, \mathrm{d}V' \end{aligned}$$

Here we employed a clever trick — differentiating $1/|\mathbf{r} - \mathbf{r}'|$ with respect to $\mathbf{r}$ is the negative of differentiating it with respect to $\mathbf{r}'$. Now that we are differentiating against $\mathbf{r}'$, we can integrate by parts to obtain

$$= -\frac{\mu_0}{4\pi} \int \left[\nabla' \cdot \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) - \frac{\nabla' \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right] \, \mathrm{d}V'.$$

The first term vanish due to being a total derivative, and the second term vanish as $\nabla \cdot \mathbf{J} = 0$ from the continuity equation, as the current is steady. Now we have the famous Biot-Savart law:

Law (Biot-Savart law)**.** The magnetic field is

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, \mathrm{d}V'.$$

If the current is localized on a curve, this becomes

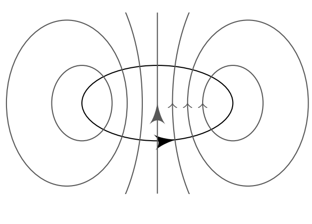
$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_C \mathrm{d}\mathbf{r}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3},$$

since $\mathbf{J}(\mathbf{r}')$ is non-zero only on the curve.

3.3 Magnetic Dipoles

According to Maxwell's equations, magnetic monopoles don't exist. However, it turns out that a localized current looks like a dipole from far far away.

Example (Current loop)**.** Take a current loop of wire C, radius R and current I.



Based on the fields generated by a straight wire, we can guess that $\mathbf{B}$ looks like this, but we want to calculate it.

By the Biot-Savart law, we know that

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}V = \frac{\mu_0 I}{4\pi} \oint_C \frac{\mathrm{d}\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}.$$

Far from the loop, $|\mathbf{r} - \mathbf{r}'|$ is small and we can use the Taylor expansion.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint_C \left(\frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \cdots \right) \mathrm{d}\mathbf{r}'.$$

Note that r is a constant of the integral, and we can take it out. The first $\frac{1}{r}$ term vanishes because it is a constant, and when we integrate along a closed loop, we get 0. So we only consider the second term. Following from green's theorem, we have:

$$\oint_C \mathbf{g} \cdot \mathbf{r}' \, \mathrm{d}\mathbf{r}' = \int_S \mathbf{g} \times \mathrm{d}\mathbf{S} = \mathbf{S} \times \mathbf{g}.$$

Using this, we have

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3},$$

where

Definition (Magnetic dipole moment)**.** The *magnetic dipole moment* is

$$\mathbf{m} = I\mathbf{S}.$$

Then

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \left(\frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} \right).$$

This is the same form as $\mathbf{E}$ for an electric dipole

Similar to electrostatics, we can do this to a general distribution to get:

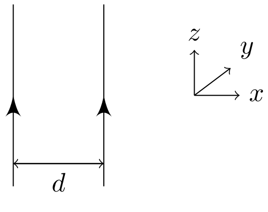
Definition (Magnetic dipole moment)**.**

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r}' \times \mathbf{J}(\mathbf{r}') \, \mathrm{d}V'.$$

3.4 Magnetic Force Examples

Now we list two classic examples to help understand the magnetic attraction/repulsion:

Example (Two parallel wires)**.**



We know that the field produced by each current is

$$\mathbf{B}_1 = \frac{\mu_0 I}{2\pi r}\hat{\varphi}.$$

The particles on the second wire will feel a force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}_1 = q\mathbf{v} \times \left(\frac{\mu_0 I_1}{2\pi d}\right)\hat{\mathbf{y}}.$$

But $\mathbf{J}_2 = nq\mathbf{v}$ and $I_2 = J_2 A$, where n is the density of particles and A is the cross-sectional area of the wire. So the number of particles per unit length is nA, and the force per unit length is

$$\mathbf{F} = na\mathbf{F} = \frac{\mu_0 I_1 I_2}{2\pi d}\hat{\mathbf{z}} \times \hat{\mathbf{y}} = -\mu_0 \frac{I_1 I_2}{2\pi d}\hat{\mathbf{x}}.$$

So if $I_1 I_2 > 0$, ie. the currents are in the same direction, the force is attractive. Otherwise the force is repulsive.

Example (General force)**.** In the general case, following the same procedure, we have, using Biot-Savart's law:

$$\mathbf{F} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \mathrm{d}\mathbf{r}_2 \times \left(\mathrm{d}\mathbf{r}_1 \times \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}\right).$$

For well-separated currents, approximated by $\mathbf{m}_1$ and $\mathbf{m}_2$, we claim that the force can be written as

$$\mathbf{F} = \frac{\mu_0}{4\pi}\nabla\left(\frac{3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}}) - (\mathbf{m}_1 \cdot \mathbf{m}_2)}{r^3}\right),$$

No, we are not going to prove this. It is a complicated mess of indices.

4 Electrodynamics

Now we look at fields that do something slightly more interesting. They change with time!

4.1 Induction

If we integrate the Maxwell equation below

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0.$$

Through a surface, use Stokes' theorem, then we have:

$$\int_C \mathbf{E} \cdot \mathrm{d}\mathbf{r} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}.$$

Definition (Electromotive force (emf))**.** The *electromotive force* (emf) is

$$\mathcal{E} = \int_C \mathbf{E} \cdot \mathrm{d}\mathbf{r}.$$

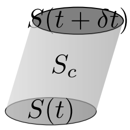
Note. This is NOT a force! This is really the voltage of the system, or the work done on a unit charge moving along the curve. Hey, don't look at me. I didn't invent the definition.

Definition (Magnetic flux and Faraday's Law)**.** The *magnetic flux* is $\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$., so the equation above becomes *Faraday's Law of induction*:

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}.$$

Note. Minus is natural. That means the magnetic field generated by the current induced opposes the original magnetic field, which kinds of prevent run off. Or the world exploding.

Also this formula still works in the most general case, when the curve is not constant: Consider a moving loop $C(t)$ bounding a surface $S(t)$. As the curve moves, the curve sweeps out a cylinder S_c.



The change in flux is

$$\begin{aligned}
\Phi(t+\delta t) - \Phi(t) &= \int_{S(t+\delta t)} \mathbf{B}(t+\delta t) \cdot \mathrm{d}\mathbf{S} - \int_{S(t)} \mathbf{B}(t) \cdot \mathrm{d}\mathbf{S} \\
&= \int_{S(t+\delta t)} \left(\mathbf{B}(t) + \frac{\partial \mathbf{B}}{\partial t}\right) \cdot \mathrm{d}\mathbf{S} - \int_{S(t)} \mathbf{B}(t) \cdot \mathrm{d}\mathbf{S} + O(\delta t^2) \\
&= \delta t \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} + \left[\int_{S(t+\delta t)} - \int_{S(t)}\right] \mathbf{B}(t) \cdot \mathrm{d}\mathbf{S} + O(\delta t^2)
\end{aligned}$$

We know that $S(t+\delta t)$, $S(t)$ and S_c together form a closed surface. Since $\nabla \cdot \mathbf{B} = 0$, the divergence of B is 0, so by the Divergence theorem the integral of $\mathbf{B}$ over a closed surface is 0. So we obtain

$$\left[\int_{S(t+\delta t)} - \int_{S(t)}\right] \mathbf{B}(t) \cdot \mathrm{d}\mathbf{S} + \int_{S_c} \mathbf{B}(t) \cdot \mathrm{d}\mathbf{S} = 0.$$

Hence we have

$$\Phi(t+\delta t) - \Phi(t) = \delta t \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} - \int_{S_c} \mathbf{B}(t) \cdot \mathrm{d}\mathbf{S}$$

We can simplify the integral over S_c by writing the surface element as

$$\mathrm{d}\mathbf{S} = (\mathrm{d}\mathbf{r} \times \mathbf{v})\, \delta t.$$

Then $\mathbf{B} \cdot \mathrm{d}\mathbf{S} = \delta t(\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{r}$. So

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \lim_{\delta \to 0} \frac{\delta \Phi}{\delta t} = \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} - \int_{C(t)} (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{r}.$$

From Maxwell's equation, we know that $\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$. So we have

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\int_C (\mathbf{E} + \mathbf{v} \times \mathbf{B})\ \mathrm{d}\mathbf{r}.$$

Now defining the emf properly we have:

$$\mathcal{E} = -\frac{\partial \Phi}{\partial t}$$

4.2 Magnetostatic Energy

This work done is identified with the energy stored in the system. Recall that the vector potential $\mathbf{A}$ is given by $\mathbf{B} = \nabla \times \mathbf{A}$. So

$$\begin{aligned} U = \frac{1}{2} I \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \frac{1}{2} I \int_S (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} &= \frac{1}{2} I \oint_C \mathbf{A} \cdot \mathrm{d}\mathbf{r} \\ &= \frac{1}{2} \int_{\mathbb{R}^3} \mathbf{J} \cdot \mathbf{A}\ \mathrm{d}V \end{aligned}$$

Using Maxwell's equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, we obtain

$$\begin{aligned} &= \frac{1}{2\mu_0} \int (\nabla \times \mathbf{B}) \cdot \mathbf{A}\ \mathrm{d}V \\ &= \frac{1}{2\mu_0} \int [\nabla \cdot (\mathbf{B} \times \mathbf{A}) + \mathbf{B} \cdot (\nabla \times \mathbf{A})]\ \mathrm{d}V \end{aligned}$$

Assuming that $\mathbf{B} \times \mathbf{A}$ vanishes sufficiently fast at infinity, the integral vanishes. So we are left with

$$= \frac{1}{2\mu_0} \int \mathbf{B} \cdot \mathbf{B}\ \mathrm{d}V.$$

Thus, in general, the energy stored in $\mathbf{E}$ and $\mathbf{B}$ is

$$U = \int \left(\frac{\varepsilon_0}{2} \mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0} \mathbf{B} \cdot \mathbf{B} \right)\ \mathrm{d}V.$$

Note this kind of is hand-waving, because we don't know if weird stuff is happening when $\mathbf{E}$ and $\mathbf{B}$ are together. Turns out there isn't, but still, be careful.

4.3 Resistance

What is the most important law in high school electrical physics?

Law (Ohm's law)**.**

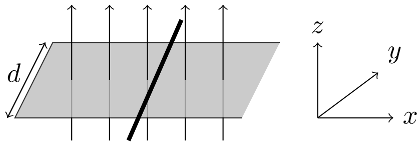
$$\mathcal{E} = IR \qquad \text{or} \qquad \mathbf{J} = \sigma \mathbf{E}.$$

Where R is defined as the *resistance.*

Definition (Resistivity and conductivity)**.** For the wire of length L and cross-sectional area A, we define the *resistivity* to be $\rho = \frac{AR}{L}$, and the conductivity $\sigma = \frac{1}{\rho}$.

Let's do a classic high school example:

Example.



Suppose the bar moves to the left with speed v. Suppose that the sliding bar has resistance R, and everything else has no resistance. You can see we assume this so that the resistance does not change with time.

There are two dynamical variables, the position of the bar $x(t)$, and the current $I(t)$.

If a current $I = q\mathbf{v}$ flows, using the Lorentz force formula, the force on the bar is

$$\mathbf{F} = IB\ell\hat{\mathbf{x}}.$$

So

$$m\ddot{x} = IB\ell.$$

We can compute the current using

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -B\ell\dot{x}.$$

So Ohm's law gives

$$IR = -B\ell\dot{x}.$$

Hence

$$m\ddot{x} = -\frac{B^2\ell^2}{R}\dot{x}.$$

Integrating once gives

$$\dot{x}(t) = -ve^{-B^2\ell^2t/mR}.$$

With resistance, we need to do work to keep a constant current. In time δt, the work needed is

$$\delta W = \mathcal{E}I\delta t = I^2R\delta t$$

using Ohm's law. So

Definition (Joule heating)**.** *Joule heating* is the energy lost in a circuit due to friction. It is given by

$$\frac{\mathrm{d}W}{\mathrm{d}t} = I^2R.$$

4.4 Displacement Currents

Let's have a quick note about the last Maxwell equation:

$$\nabla \times \mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial \mathbf{E}}{\partial t}\right)$$

Now what is $\mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t}$? This was really discovered mathematically to make the equations consistent with charge conservation and is called a *displacement current.*

4.5 EM Waves. Let there be light!^{TM}

Now we let $\rho = 0$ and $J = 0$ to find solutions to Maxwell's equations in vacuum. We differentiate the fourth equation wrt t and use $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$:

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \nabla \times \frac{\partial \mathbf{B}}{\partial t} = \nabla(\underbrace{\nabla \cdot \mathbf{E}}_{=\rho/\varepsilon_0 = 0}) + \nabla^2 \mathbf{E} = \nabla^2 \mathbf{E}.$$

Similarly, we have:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0,$$

This is clearly the wave equation, with the speed of the wave being $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$, the speed of light! Now the most important solutions are the *monochromatic waves* with:

$$\mathbf{E} = (0, E_0 \sin(kx - \omega t), 0).$$

Where we assume the electric field is oscillating in the y direction, and the wave is traveling in the x direction. Equivalently, we could've had a wave with electric field oscillating in z direction.

Definition (Amplitude, wave number and frequency)**.**

(i) E_0 is the *amplitude*

(ii) k is the *wave number.* $\lambda = \frac{2\pi}{k}$ is the wavelength.

(iii) ω is the *(angular) frequency.*

Since the wave travels at speed c, we have:

$$\omega^2 = c^2 k^2$$

Now we use $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ to solve for B, giving:

$$B = \frac{E_0}{c}\sin(kx - \omega t).$$

Now we make out solves easier, in the most general form we would write:

$$\mathbf{E} = \mathbf{E}_0 \exp(i(\mathbf{k} \cdot \mathbf{x} - \omega t)), \qquad \mathbf{B} = \mathbf{B}_0 \exp(i(\mathbf{k} \cdot \mathbf{x} - \omega t)),$$

where $\mathbf{k}$ is the wave vector, or the direction in which the wave is travelling. $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{k}$ are orthogonal to each other.

Note. THIS IS VERY IMPORTANT! The complex notation is for *notation* only. The only part that actually exists is the real part, and before we do anything, we need to take the real part of it.

Definition (Polarization)**.** A solution with real $\mathbf{E}_0, \mathbf{B}_0, \mathbf{k}$ is said to be *linearly polarized*, which means waves oscillate up and down in a fixed plane. If $\mathbf{E}_0 = \alpha + \beta i$ and $\mathbf{B}_0$ are complex, then it is said to be *elliptically polarized.* In the special case where $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$ and $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0$, this is *circular polarization.*

This says that the waves oscillate up and down in a fixed plane.

Now if we have a wave $\mathbf{E}_{\text{inc}} = E_0 \hat{\mathbf{y}} \exp(i(kx + \omega t))$, coming at right angle into a conductor, then we know since $\mathbf{E} = 0$ inside the conductor, this solution clearly doesn't work. Using the image method again, we have, with $\mathbf{E}_{\text{ref}} = -E_0 \hat{\mathbf{y}} \exp(i(-kx - \omega t))$.:

$$\mathbf{E} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{ref}}.$$

Where the reflected wave is caused by the surface current. Why?, first let's derive the incident and reflected magnetic field:

$$\mathbf{B}_{\text{inc}} = \frac{E_0}{c} \hat{\mathbf{z}} \exp(i(kx - \omega t)) \qquad \mathbf{B}_{\text{ref}} = \frac{E_0}{c} \hat{\mathbf{z}} \exp(i(-kx - \omega t))$$

Then the surface magnetic field is:

$$\mathbf{B} \cdot \hat{\mathbf{z}}|_{x=0^-} = \frac{2E_0}{c} e^{-i\omega t},$$

This gives rise to a surface current, as inside the conductor we have $\mathbf{B} = 0$. The surface current is, by the same method as in Ampere's law:

$$\mathbf{B} \cdot \hat{\mathbf{z}}|_{x=0^-} = \frac{2E_0}{c} e^{-i\omega t},$$

The surface current is oscillating, so charges are accelerating, and they give out light in the process, which are the reflected waves. And that's why metals are shiny!

4.6 Poynting Vector

Now we would derive Poynting's theorem as followed:

First we differentiate the energy equation for an electric field $\mathbf{E}$ and $\mathbf{B}$:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\int_V \mathbf{J} \cdot \mathbf{E} \,\mathrm{d}V - \frac{1}{\mu_0} \int_S (\mathbf{E} \times \mathbf{B}) \cdot \,\mathrm{d}\mathbf{S}.$$

From vector identities, we have $\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \mathbf{B} \cdot (\nabla \times \mathbf{E}) = \nabla \cdot (\mathbf{E} \times \mathbf{B})$:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\int_V \mathbf{J} \cdot \mathbf{E} \,\mathrm{d}V - \frac{1}{\mu_0} \int_S (\mathbf{E} \times \mathbf{B}) \cdot \,\mathrm{d}\mathbf{S}.$$

So:

Theorem (Poynting theorem)**.**

$$\underbrace{\frac{\mathrm{d}U}{\mathrm{d}t} + \int_V \mathbf{J} \cdot \mathbf{E} \,\mathrm{d}V}_{\text{Total change of energy in } V \text{ (fields + particles)}} = \underbrace{-\frac{1}{\mu_0} \int_S (\mathbf{E} \times \mathbf{B}) \cdot \mathrm{d}S}_{\text{Energy that escapes through the surface } S}.$$

Definition (Poynting vector)**.** The *Poynting vector* is

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.$$

Note. The poynting vector can and should be seen as the energy per unit surface area geiven into the system at any given time. It characterizes the energy transfer.

5 Electromagnetism and Relativity

Now the original course ended here. But our great David Tong (rightfully?) decided it was a bit short and added this section. Now, first things first:

5.1 Vectors and Covectors

Wait....where was this in Special Relativity? Don't panic, we will go through this again. First off we had the Minkowski metric η, vectors X^μ, Y^μ and a dot product:

$$X \cdot Y = X^T \eta Y.$$

Right? Now we define $X_\mu = \eta_{\mu v} X^v$ and $X^\mu = X$. So the dot product now becomes $X_\mu X^\mu$. Now pure mathematicians will start mumbling about that one is contravariant and one is covariant, but we will skip this part. The only thing we care is that from this point on:

- You can only sum iff the same indices appear once above and once below.
- You can use $\eta_{\mu v}$ to raise or lower indices. To raise them, use $\eta^{\mu v}$, which is the inverse matrix, but actually the same matrix.

5.1.1 Lorentz Transformations

These are the transformations or matrices that preserve the Minkowski metric. What does it mean? In Einstein notation, this means the matrix/transformation Λ^μ_ν satisfy:

$$\Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\nu = \eta_{\mu\nu}.$$

Ok, wait a sec... didn't we just use 2 lower or upper indices to represent a matrix? Why are we using one upper and one lower now? Hold on that thought, we will come to it very soon.

From special relativity, we know there are two classes of Lorentz transformations:

(i) Rotations:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}$$

where $R^T R = 1$, ie. is an orthogonal matrix.

(ii) Boosts, eg. a boost in the x direction is

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now thus, the lower indices one transform as:

$$X_\mu \mapsto X'_\mu = \eta_{\mu\nu} X'^\nu = \eta_{\mu\nu} \Lambda^\nu{}_\sigma X^\sigma = \eta_{\mu\nu} \Lambda^\nu{}_\sigma \eta^{\sigma\rho} X_\rho = \Lambda_\mu{}^\rho X_\rho$$

where $\Lambda_\mu{}^\rho$, as you have probably guessed, is the inverse of $\Lambda^\mu{}_\rho$. But also remember, it is also the transpose of $\Lambda^\rho{}_\mu$, as it is an orthogonal matrix (in the Minkowski metric).

Definition (Vectors and co-vectors)**.** *Vectors* have indices up and transform according to $X^\mu \mapsto \Lambda^\mu{}_\nu X^\nu$.

Co-vectors have indices down and transform according to $X_\mu \mapsto \Lambda_\mu{}^\nu X_\nu$.

We can explore different objects and see if they are vectors or co-vectors.

Definition (4-derivative)**.** The relativistic generalization of ∇ is the *4-derivative*, defined to be

$$\partial_\mu = \frac{\partial}{\partial X^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right).$$

5.2 Vectors and Covectors

This is the "official" explanation of how the indices actually work:

Definition (Vectors and co-vectors)**.** *Vectors* have indices up and transform according to $X^\mu \mapsto \Lambda^\mu{}_\nu X^\nu$.

Co-vectors have indices down and transform according to $X_\mu \mapsto \Lambda_\mu{}^\nu X_\nu$.

Now vectors are just the normal vectors we know. ONce we changed the basis, we need to change it in the *inverse* way of the basis change to get our new vector, because our vector needs to stay the same.

But some things change in the same way as a basis, such as:

Definition (4-derivative)**.** The relativistic generalization of ∇ is the *4-derivative*, defined to be

$$\partial_\mu = \frac{\partial}{\partial X^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right).$$

This is what a *co-vector* really is, because it *co*-changes with the basis.

Here is the definition of a tensor if anyone forgot:

Definition (Tensor)**.** A *tensor* of type (m, n) is a quantity

$$T^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_n}$$

which transforms as

$$T'^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_n} = \Lambda^{\mu_1}{}_{\rho_1} \cdots \Lambda^{\mu_m}{}_{\rho_m} \Lambda_{\nu_1}{}^{\sigma_1} \cdots \Lambda_{\nu_n}{}^{\sigma_n} \times T^{\rho_1, \cdots, \rho_m}{}_{\sigma_1, \cdots, \sigma_n}.$$

5.3 Conserved Currents

We want the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

To work in 3+1D, or 4-vectors. Turns out it does, with:

$$J^\mu = \begin{pmatrix} \rho c \\ \mathbf{J} \end{pmatrix}$$

And the charge equation becomes, simply, $\partial_\mu J^\mu = 0$. Nice, huh?

5.4 Gauge potentials and EM Fields

Now we define our potentials as before:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Ok, ok...I know something has changed. The extra $-\frac{\partial A}{\partial t}$ term is to compensate the fact that when we defined the potential for **E** we had no magnetic field, but in reality we do have.Now since we have 4 components of a thing,we make it into a 4-vector!

$$A^\mu = \begin{pmatrix} \phi/c \\ \mathbf{A} \end{pmatrix}$$

Trust me this works. Now we define the anti-symmetric electromagnetic tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Since this is antisymmetric, the diagonals are all 0, and $A_{\mu\nu} = -A_{\nu\mu}$. So this thing has $(4\times4-4)/2 = 6$ independent components. And this encapsulates all information about the EM field. Writing it out, we have:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

Note this is also invariant under gauge transformations.

Under a boost by v in the x-direction, we have:

$$E'_x = E_x$$
$$E'_y = \gamma(E_y - vB_z)$$
$$E'_z = \gamma(E_z + vB_y)$$
$$B'_x = B_x$$
$$B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$
$$B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

By multiplying the Lorenz matrix with this one. We look at an example to show the relationship between **E** and **B** fields:

Example (Boosted line charge)**.** An infinite line along the x direction with uniform charge per unit length, η, has electric field

$$\mathbf{E} = \frac{\eta}{2\pi\varepsilon_0(y^2+z^2)}\begin{pmatrix} 0 \\ y \\ z \end{pmatrix}.$$

The magnetic field is $\mathbf{B} = 0$. Plugging this into the equation above, an observer in frame S' boosted with $\mathbf{v} = (v, 0, 0)$, ie. parallel to the wire, sees

$$\mathbf{E} = \frac{\eta\gamma}{2\pi\varepsilon_0(y^2+z^2)}\begin{pmatrix} 0 \\ y \\ z \end{pmatrix} = \frac{\eta\gamma}{2\pi\varepsilon_0(y'^2+z'^2)}\begin{pmatrix} 0 \\ y' \\ z' \end{pmatrix}.$$

Also,

$$\mathbf{B} = \frac{\eta\gamma v}{2\pi\varepsilon_0\sigma^2(y^2+z^2)}\begin{pmatrix}0\\ z\\ -y\end{pmatrix} = \frac{\eta\gamma v}{2\pi\varepsilon_0\sigma^2(y'^2+z'^2)}\begin{pmatrix}0\\ z'\\ -y'\end{pmatrix}.$$

In frame S', the charge density is Lorentz contracted to $\gamma\eta$. The magnetic field can be written as

$$\mathbf{B} = \frac{\mu_0 I'}{2\pi\sqrt{y'^2+z'^2}}\,\hat{\varphi}',$$

where $\hat{\varphi}' = \frac{1}{\sqrt{y'^2+z'^2}}\begin{pmatrix}0\\ -z'\\ y'\end{pmatrix}$ is the basis vector of cylindrical coordinates, and $I' = -\gamma\eta v$ is the current.

This is what we calculated from Ampere's law previously. But we didn't *use* Ampere's law here. We used Gauss' law, and then applied a Lorentz boost.

We see that magnetic fields are relativistic effects of electric fields. They are what we get when we apply a Lorentz boost to an electric field. So relativity is not only about *very fast objects*™. It is there when you stick a magnet onto your fridge!

Example (Boosted point charge)**.** A point charge Q, at rest in frame S has

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} = \frac{Q}{4\pi\varepsilon_0^2(x^2+y^2+z^2)^{3/2}}\begin{pmatrix}x\\ y\\ z\end{pmatrix},$$

and $\mathbf{B} = 0$.

In frame S', boosted with $\mathbf{v} = (v, 0, 0)$, we have

$$\mathbf{E}' = \frac{Q}{4\pi\varepsilon_0(x^2+y^2+z^2)^{3/2}}\begin{pmatrix}x\\ \gamma y\\ \gamma z.\end{pmatrix}.$$

We need to express this in terms of x', y', z', instead of x, y, z. Then we get

$$\mathbf{E}' = \frac{Q\gamma}{4\pi\varepsilon_0(\gamma^2(x'+vt')^2+y'^2+z'^2)}\begin{pmatrix}x'+vt'\\ y'\\ z'\end{pmatrix}.$$

Suppose that the particle sits at $(-vt', 0, 0)$ in S'. Let's look at the electric at $t' = 0$. Then the radial vector is $\mathbf{r}' = \begin{pmatrix}x'\\ y'\\ z'\end{pmatrix}$. In the denominator, we write

$$\begin{aligned}\gamma^2(x'^2+y'^2+z'^2) &= (\gamma^2-1)x'^2 + r'^2\\ &= \frac{v^2\gamma^2}{c^2}x'^2 + r'^2\\ &= \left(\frac{v^2\gamma^2}{c}^2\cos^2\theta + 1\right)r'^2\\ &= \gamma^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)r'^2\end{aligned}$$

where θ is the angle between the x' axis and $\mathbf{r}'$.

So

$$\mathbf{E}' = \frac{1}{\gamma^2\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}'.$$

The factor $1/\gamma^2\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}$ squashes the electric field in the direction of motion. So what we did previously, which would not give the factor is wrong! Well, kind of. Notice the factor is very close to 1 if speeds are small.

There is also a magnetic field

$$\mathbf{B} = \frac{\mu_0 Q\gamma}{4\pi(\gamma^2(x'+vt')^2+y'^2+z'^2)^{3/2}}\begin{pmatrix}0\\ z'\\ -y'\end{pmatrix}.$$

5.4.1 Lorenz Invariants

Now we want something that everyone agrees on. The first thing is:

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = -\frac{\mathbf{E}^2}{c^2}+\mathbf{B}^2,$$

Then we define the *dual electromagnetic tensor*, which is:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. = \begin{pmatrix}0 & -B_x & -B_y & -B_z\\ B_x & 0 & E_z/c & -E_y/c\\ B_y & -E_z/c & 0 & E_x/c\\ B_z & E_y/c & -E_x/c & 0\end{pmatrix}.$$

Where $\varepsilon^{\mu\nu\rho\sigma}$ is the natural extension of the Levi-Civita symbol into 4 dimensions. The half comes from the fact that flipping the indices gives 2 times the contribution. This is also a tensor and we have

$$\frac{1}{4}\tilde{F}^{\mu\nu}F_{\mu\nu} = \mathbf{E}\cdot\mathbf{B}/c.$$

5.5 Maxwell Equations

Now. Show time. The Maxwell equations relativistically are:

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$
$$\partial_\mu \tilde{F}^{\mu\nu} = 0.$$

Wait...thats it? Yup. Well, first one should notice each equation is 4 equations, but this is it. We can check that we can actually recover the original equations, but we will not. Avoid algebra when you can.

5.6 The Lorentz Force Law

The final aspect of electromagnetism is the Lorentz force law for a particle with charge q moving with velocity $\mathbf{u}$:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}).$$

To write this in relativistic form, we use the proper time τ (time experienced by particle), which obeys

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma(\mathbf{u}) = \frac{1}{\sqrt{1 - u^2/c^2}}.$$

We define the 4-velocity $U = \frac{\mathrm{d}X}{\mathrm{d}\tau} = \gamma \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}$, and 4-momentum $P = \begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix}$, where E is the energy. Note that E is the energy while $\mathbf{E}$ is the electric field.

The Lorentz force law can be written as

$$\frac{\mathrm{d}P^\mu}{\mathrm{d}\tau} = qF^{\mu\nu}U_\nu.$$

We show that this does give our original Lorentz force law:

When $\mu = 1, 2, 3$, we obtain

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = q\gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

By the chain rule, since $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma$, we have

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

So the good, old Lorentz force law returns. Note that here $\mathbf{p} = m\gamma\mathbf{v}$, the relativistic momentum, not the ordinary momentum.

But how about the $\mu = 0$ component? We get

$$\frac{\mathrm{d}P^0}{\mathrm{d}\tau} = \frac{1}{c}\frac{\mathrm{d}E}{\mathrm{d}\tau} = \frac{q}{c}\gamma\mathbf{E} \cdot \mathbf{v}.$$

This says that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = q\mathbf{E} \cdot \mathbf{v},$$

which is our good old formula for the work done by an electric field.

Example (Motion in a constant field)**.** Suppose that $\mathbf{E} = (E, 0, 0)$ and $\mathbf{u} = (v, 0, 0)$. Then

$$m\frac{\mathrm{d}(\gamma u)}{\mathrm{d}t} = qE.$$

So

$$m\gamma u = qEt.$$

So

$$u = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{qEt}{\sqrt{m^2 + q^2E^2t^2/c^2}}.$$

Note that $u \to c$ as $t \to \infty$. Then we can solve to find

$$x = \frac{mc^2}{q}\left(\sqrt{1 + \frac{q^2E^2t^2}{mc^2}} - 1\right)$$

For small t, $x \approx \frac{1}{2}qEt^2$, as expected.

Example (Motion in constant magnetic field)**.** Suppose $\mathbf{B} = (0, 0, B)$. Then we start with

$$\frac{\mathrm{d}P^0}{\mathrm{d}\tau} = 0 \Rightarrow E = m\gamma c^2 = \text{constant}.$$

So $|\mathbf{u}|$ is constant. Then

$$m\frac{\partial(\gamma\mathbf{u})}{\partial t} = q\mathbf{u} \times \mathbf{B}.$$

So

$$m\gamma\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = q\mathbf{u} \times \mathbf{B}$$

since $|\mathbf{u}|$, and hence γ, is constant. This is the same equation we saw in Dynamics and Relativity for a particle in a magnetic field, except for the extra γ term. Therefore the particle goes in circles with frequency

$$\omega = \frac{qB}{m\gamma}.$$