Math Tripos Part IA: Differential Equations

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Basic calculus

Informal treatment of differentiation as a limit, the chain rule, Leibnitz's rule, Taylor series, informal treatment of O and o notation and l'Hôpitals rule; integration as an area, fundamental theorem of calculus, integration by substitution and parts. [3]

Informal treatment of partial derivatives, geometrical interpretation, statement (only) of symmetry of mixed partial derivatives, chain rule, implicit differentiation. Informal treatment of differentials, including exact differentials. Differentiation of an integral with respect to a parameter. [2]

First-order linear differential equations

Equations with constant coefficients: exponential growth, comparison with discrete equations, series solution; modelling examples including radioactive decay.

Equations with non-constant coefficients: solution by integrating factor. [2]

Nonlinear first-order equations

Separable equations. Exact equations. Sketching solution trajectories. Equilibrium solutions, stability by perturbation; examples, including logistic equation and chemical kinetics. Discrete equations: equilibrium solutions, stability; examples including the logistic map. [4]

Higher-order linear differential equations

Complementary function and particular integral, linear independence, Wronskian (for second-order equations), Abel's theorem. Equations with constant coefficients and examples including radioactive sequences, comparison in simple cases with difference equations, reduction of order, resonance, transients, damping. Homogeneous equations. Response to step and impulse function inputs; introduction to the notions of the Heaviside step-function and the Dirac delta-function. Series solutions including statement only of the need for the logarithmic solution. [8]

Multivariate functions: applications

Directional derivatives and the gradient vector. Statement of Taylor series for functions on $\mathbf{R}^n$. Local extrema of real functions, classification using the Hessian matrix. Coupled first order systems: equivalence to single higher order equations; solution by matrix methods. Non-degenerate phase portraits local to equilibrium points; stability.

Simple examples of first- and second-order partial differential equations, solution of the wave equation in the form $f(x + ct) + g(x - ct)$. [5]

Contents

1 Differentiation

1.1 Differentiation

Definition. The *derivative* of a function fx w.r.t to an independent variable x is defined as the rate of change of $f(x)$ with x, or symbolically:

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x)$ is differentiable at a point $x = a$ iff the limit exists. ($|x|$ is not differentiable at $x = 0$)

Note. $\frac{df}{dx}, f'(x), \frac{d}{dx}f(x)$ all mean the same thing. Higher derivatives follow the same rule ($\frac{d^2 f}{dx^2} = f''(x) = \frac{d}{dx}(\frac{d}{dx}f(x))$). However, f' represents the derivative w.r.t *the argument*.

1.2 Small o and big O notations

Definition.

(i) $f(x) = o(g(x))$ as $x \to x_0$ if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$, so $f(x)$ is much smaller than $g(x)$.

(ii) $f(x) = O(g(x))$ as $x \to x_0$ if $\frac{f(x)}{g(x)}$ is *bounded* as $x \to x_0$, so $f(x)$ is about the size of $g(x)$.

Note. *Bounded* doesn't mean limit exists. $f(x) = o(g(x)) \Rightarrow f(x) = O(g(x))$

Corollary. $f(x_0 + h) = f(x_0) + f'(x_0)h + o(h)$

Proof.

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} + \frac{0(h)}{h}$$

from definition of the derivative and the small o notation. Result follows. □

1.3 Methods of Differentiation

Theorem (Chain rule)**.** Given $f(x) = F(g(x))$, then

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}F}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}.$$

Proof. Assuming that $\frac{\mathrm{d}g}{\mathrm{d}x}$ exists and is therefore finite, we have

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h \to 0} \frac{F(g(x+h)) - F(g(x))}{h} \\
&= \lim_{h \to 0} \frac{F(g(x)) + (hg'(x) + o(h))F'(g(x)) + o(hg'(x) + o(h)) - F(g(x))}{h} \\
&= g'(x)F'(g(x)) = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}
\end{aligned}$$

□

Theorem (Leibinz's Rule)**.** Given $f = uv$, then

$$f^{(n)}(x) = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)}$$

where $f^{(n)}$ is the n-th derivative of f.

1.4 L'Hopital's Rule

Theorem. If $f(x), g(x)$ differentiable at x_0, and $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$, then:

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Proof. From the Taylor's Theorem, we have $f(x) = f(x_0) + (x - x_0)f'(x_0) + o(x - x_0)$, and similarly for $g(x)$. Thus

$$\begin{aligned}
\lim_{x\to x_0} \frac{f(x)}{g(x)} &= \lim_{x\to x_0} \frac{f(x_0) + (x - x_0)f'(x_0) + o(x - x_0)}{g(x_0) + (x - x_0)g'(x_0) + o(x - x_0)} \\
&= \lim_{x\to x_0} \frac{f'(x_0) + \frac{o(x-x_0)}{x-x_0}}{g'(x_0) + \frac{o(x-x_0)}{x-x_0}} \\
&= \lim_{x\to x_0} \frac{f'(x)}{g'(x)}
\end{aligned}$$

□

2 Integration

2.1 Integration

Definition. An *integral* is a limit of an (infinite) sum, or the "area under the graph":

$$\int_a^b f(x)\, \mathrm{d}x = \lim_{\Delta x\to 0} \sum_{n=0}^{N} f(x_n)\Delta x.$$

If f is differentiable, the total area under the graph from a to b is:

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} (f(x_n)\Delta x) + N \cdot O(\Delta x^2) = \lim_{N\to\infty} \sum_{n=0}^{N-1} (f(x_n)\Delta x) + O(\Delta x) = \int_a^b f(x)\, \mathrm{d}x$$

due to the missing "triangle"$O(\Delta x^2)$ of the area in the sum.

Theorem (Fundamental Theorem of Calculus)**.** Let $F(x) = \int_a^x f(t)\, \mathrm{d}t$. Then $F'(x) = f(x)$.

Proof.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}F(x) &= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\ \mathrm{d}t\\ &= \lim_{h\to 0}\frac{1}{h}[f(x)h + O(h^2)]\\ &= f(x)\end{aligned}$$

□

Similarly, we have $\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b f(t)\ \mathrm{d}t = -f(x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{g(x)} f(t)\ \mathrm{d}t = f(g(x))g'(x)$.

Note. We write $\int f(x)\ \mathrm{d}x = \int^x f(t)\ \mathrm{d}t$,; the unspecified lower limit gives constant $+C$

2.2 Methods of integration

Useful Hints for trig substitution:

Useful identity	Part of integrand	Substitution
$\cos^\theta + \sin^\theta = 1$	$\sqrt{1-x^2}$	$x = \sin\theta$
$1+\tan^2\theta = \sec^2\theta$	$1+x^2$	$x = \tan\theta$
$\cosh^2 u - \sinh^2 u = 1$	$\sqrt{x^2-1}$	$x = \cosh u$
$\cosh^2 u - \sinh^2 u = 1$	$\sqrt{1-x^2}$	$x = \sinh u$
$1-\tanh^2 u = sech^2 u$	$1-x^2$	$x = \tanh u$

2.3 Integration by Parts

Integrating the product rule, we have:

$$\int uv'\ \mathrm{d}x = uv - \int vu'\ \mathrm{d}x.$$

Example 1 (THE example for Integration by Parts)**.** Consider $\int \log x\ \mathrm{d}x$. Let $u = \log x$ and $v' = 1$. Then $u' = \frac{1}{x}$ and $v = x$. So we have

$$\begin{aligned}\int \log x\ \mathrm{d}x &= x\log x - \int \mathrm{d}x\\ &= x\log x - x + C\end{aligned}$$

3 Partial Differentiation

3.1 partial Differentiation

Definition (Partial derivative)**.** Given a function of several variables $f(x, y)$, the *partial derivative* of f with respect to x is the rate of change of f as x varies, keeping y constant.

$$\left.\frac{\partial f}{\partial x}\right|_y = \lim_{\delta x\to 0}\frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

Example 2. Consider $f(x,y) = x^2 + y^3 + e^{xy^2}$. Just treat all other variables held constant as *constants*, e.g.:

$$\left.\frac{\partial f}{\partial y}\right|_y = 2x + y^2 e^{xy^2}.$$

Note. If the variables to be kept constant are not given, e.g. simply $\frac{\partial f}{\partial x}$, it is assumed that *ALL* other variables are being held constant. We also write $f_x = \frac{\partial f}{\partial x}$ and $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$. Also, $f_{xy} = f_{yx}$.

3.2 Chain rule

Consider an arbitrary displacement in any direction $(x,y) \to (x+\delta x, y+\delta y)$. We have

$$\begin{aligned}
\delta f &= f(x+\delta x, y+\delta y) - f(x,y) \\
&= f(x+\delta x, y+\delta y) - f(x+\delta x, y) + f(x+\delta x, y) - f(x,y) \\
&= f_y(x+\delta x, y)\delta y + o(\delta y) + f_x(x,y)\delta x + o(\delta x) \\
&= (f_y(x,y) + O(\delta x))\delta y + o(\delta y) + f_x(x,y)\delta x + o(\delta x)
\end{aligned}$$

Take the limit as $\delta x, \delta y \to 0$, we have

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y.$$

Note. We use "d" as differentials, knowing we will sum (integral) or divide by another infinitesimal before evaluating.

3.3 Implicit Differentiation

Consider the contour surface of a function $F(x,y,z)$ given by $F(x,y,z) = \text{const}$. This implicitly defines $z = z(x,y)$. e.g. If $f(x,y,z) = xy^2 + yz^2 + z^5 x = 5$, then we can have $x = \frac{y - yz^2}{y^2 + z^5}$. Even though $z(x,y)$ is insolvable due to quintic equation, derivatives of $z(x,y)$ can still be found by differentiation $F(x,y,z) = \text{const}$ w.r.t. x holding y constant. e.g.

$$\begin{aligned}
\frac{\partial}{\partial x}(xy^2 + yz^2 + z^5 x) &= \frac{\partial}{\partial x}5 \\
\frac{\partial z}{\partial x} &= -\frac{y^2 + z^5}{2yz + 5z^4 x}
\end{aligned}$$

Theorem (Multivariable implicit differentiation)**.** Given an equation

$$F(x,y,z) = c$$

for some constant c, we have

$$\left.\frac{\partial z}{\partial x}\right|_y = -\frac{(\partial F)/(\partial x)}{(\partial F)/(\partial z)}$$

Proof.

$$\left.\frac{\partial F}{\partial x}\right|_y = \frac{\partial F}{\partial x}\left.\frac{\partial x}{\partial x}\right|_y + \frac{\partial F}{\partial y}\left.\frac{\partial y}{\partial y}\right|_y + \frac{\partial F}{\partial z}\left.\frac{\partial z}{\partial x}\right|_y = 0$$
$$\left.\frac{\partial z}{\partial x}\right|_y = -\frac{(\partial F)/(\partial x)}{(\partial F)/(\partial z)}$$

□

Note. Reciprocals hold with partial derivatives as long as the variables keeping constant are the same.

3.4 Differentiation of Integral w.r.t parameter in Integrand

Consider a family of functions $f(x, c)$. Define $I(b, c) = \int_a^b f(x, c)\mathrm{d}x$. We have

$$\begin{aligned}\frac{\partial I}{\partial c} &= \lim_{\delta c \to 0} \frac{1}{\delta c}\left[\int_0^b f(x, c+\delta c)\mathrm{d}x - \int_0^b f(x, c)\, \mathrm{d}x\right] \\ &= \lim_{\delta c \to 0} \int_0^b \frac{f(x, c+\delta c) - f(x, c)}{\delta c}\, \mathrm{d}x \\ &= \int_0^b \frac{\partial f}{\partial c}\, \mathrm{d}x\end{aligned}$$

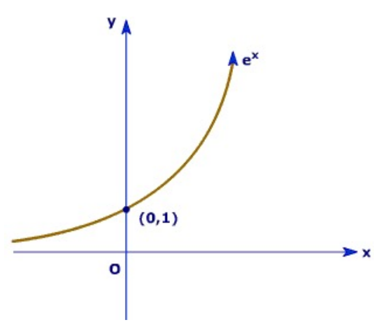
If $I(b(x), c(x)) = \int_0^{b(x)} f(y, c(x))\mathrm{d}y$, then by the chain rule, we have

$$\frac{\mathrm{d}I}{\mathrm{d}x} = \frac{\partial I}{\partial b}\frac{\mathrm{d}b}{\mathrm{d}x} + \frac{\partial I}{\partial c}\frac{\mathrm{d}c}{\mathrm{d}x} = f(b, c)b'(x) + c'(x)\int_0^b \frac{\partial f}{\partial c}\, \mathrm{d}y.$$

4 First-order differential equation

4.1 The Exponential Function

Consider $f(x) = a^x$, where $a > 0$ is constant.



$$\frac{df}{dx} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h} = \lambda a^x = \lambda f(x)$$

Where

$$\lambda = \lim_{h \to 0} \frac{a^h - 1}{h} = const = f'(0)$$

We define the function $f(x) = e^x$ by $\frac{df}{dx} = f(x)$ with $\lambda = 1$ with $f(0) = 1$. We can prove that $e = \lim_{k \to \infty} (1 + \frac{1}{k})^k$. So, If $y = a^x = e^{x \ln a}$ then $\frac{dy}{dx} = (\ln a)e^{x \ln a} = (\ln a)a^x \Rightarrow \lambda = \ln a$.

Note. $\ln x \equiv \log_e x \equiv \log x$

4.2 First-order Linear Homogenous Ordinary Differential Equation

Definition. An *eigenfunction* under an operator is a function that only differs by a constant after applying the operator.

Example 3. $\frac{d}{dx}(e^{mx}) = m(e^{mx})$ so e^{mx} is an *eigenfunction* of the differential operator of $\frac{d}{dx}$.

Theorem. Any linear, homogeneous, ordinary differential equation with constant coefficient has solutions of the form e^{mx}. *Linear* means the dependent variable appears linearly (no powers). *Homogeneous* means $y = 0$ is a solution. *constant coefficients* means the independent variable does not appear explicitly.

(i) Because equations are linear and homogeneous, any multiple of a solution is also a solution. So $y = Ae^{mx}$ is a solution for any A.

(ii) An nth-order *linear* differential equation has *only* n independent solutions, so $y = Ae^{\frac{3x}{5}}$ is the general solution of the solution $5y' - 3y = 0$.

(iii) Any solution in nth-order linear differential equation is a linear combination of eigenfunctions.

Note. Determine A by applying a boundary condition at some specified value of x (usually $x = 0$)

4.2.1 Discrete equation

Example 4. $5y' - 3y = 0$, $y = y_0$ at $x = 0$

Approximate by $5\frac{y_{n+1} - y_n}{h} - 3y_n = 0 \Rightarrow y_{n+1} \approx (1 + \frac{3h}{5})y_n$ (compound interest)
Apply recurrence relation repeatedly

$$y_n = (1 + \frac{3h}{5})y_{n-1} = (1 + \frac{3h}{5})^n y_0$$

For a given value of x, choose $h = \frac{x}{n}$, so

$$y_n = y_0(1 + \frac{3x}{5n})^n$$

Take limit as $n \to \infty$

$$y(x) = \lim_{n \to \infty} y_o(1 + \frac{3x}{5n})^n$$

in agreement with solution of differential equation.

4.3 Forced (inhomogeneous) equations

Example 5.

(i) $5y' - 3y = 10$

(ii) $5y' - 3y = e^x$

Solution

(i) This is called *constant forcing*. We just need to guess a constant particular solution, and clearly $y_p = -\frac{10}{3}$ would do.

(ii) This is called *eigenfunction forcing*. Since e^x is an eigenfunction, we guess $y_p = Ae^x$. Solve for A to find $A = \frac{1}{2}$

4.4 Non-constant coefficients

Given form

$$y' + p(x)y = c(x)$$

We can use an integrating factor $\mu = e^{\int p \text{ dx}}$ to solve this:

$$(\mu y)' = c(x)\mu \Rightarrow y = \frac{\int c(x)\mu \text{ dx}}{\mu}$$

4.5 Nonlinear first-order equations

In a general, a first order equation has the form

$$Q(x,y)\frac{dy}{dx} + P(x,y) = 0$$

4.5.1 Separable Equations

Definition. The equation is *separable* if it can be manipulated into the form $g(y)dy = p(x)dx$

In which case the solution can be found by integrating both sides.

$$\int g(y)dy = \int p(x)dx$$

Example 6.

$$(x^2y - 3y)\frac{dy}{dx} - 2xy^2 = 4x$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x + 2xy^2}{x^2y - 3y} = \frac{2x(2+y^2)}{y(x^2-3)}$$

$$\Rightarrow \int \frac{y}{2+y^2}dy = \int \frac{2x}{x^2-3}dx$$

$$\Rightarrow (y^2+2)^{\frac{1}{2}} = A(x^2-3)$$

4.5.2 Exact Equations

$$Q(x,y)\frac{dy}{dx} + P(x,y) = 0$$

is an exact equation iff the differential form

$$Q(x,y)\ dy + P(x,y)\ dx$$

is exact. i.e. there is a function $f(x,y)$ of which the above equation is the differential $\mathrm{d}f$.

For exact equations:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y} = P(x,y)dx + Q(x,y)dy \ \Rightarrow\ \frac{\partial f}{\partial x} = P, \frac{\partial f}{\partial y} = Q$$

By taking mixed second partial derivatives, we need the following equality:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Note. If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout a simply-connected domain $\mathcal{D}$, then $P\ dx + Q\ dy$ is an exact differential of a single-valued function in $\mathcal{D}$. A domain $\mathcal{D}$ is *simply connected* if any closed curve in $\mathcal{D}$ can be shrunk to a point in $\mathcal{D}$ without leaving $\mathcal{D}$. (no holes, unlike torus)

If the equation is exact then solution f =constant can be found by integrating the

$$df = P(x,y)dx + Q(x,y)dy$$

Example 7.

$$6y(y-2)\frac{dy}{dx} + (2x - 3y^2) = 0$$

$$\Rightarrow (2x - 3y^2)dx + 6y(y-x)dy = 0$$

One could check it is exact so we have

$$f = x^2 - 3xy^2 + h(y)$$

$$\frac{\partial f}{\partial y} = -6xy + \frac{dh}{dy} = 6y^2 - 6xy \Rightarrow \frac{dh}{dy} = 6y^2$$

$$h = 2y^3 + C \Rightarrow f = x^2 - 3xy^2 + 2y^3 + C$$

Since the equation was $\mathrm{d}f = 0$:

$$x^2 - 3xy^2 + 2y^3 = const$$

4.6 Solution curves (Trajectories)

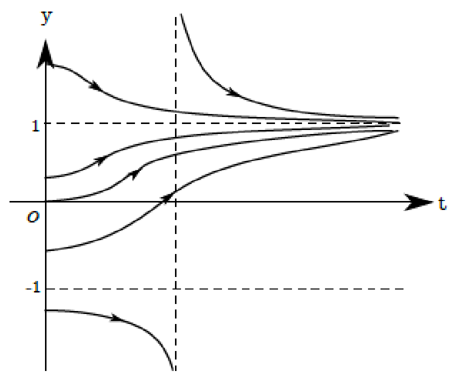
Example 8.

$$\frac{dy}{dt} = t(1 - y^2)$$

Separable equation, so after calculation, we have:

$$\Rightarrow y = \frac{A - e^{-t^2}}{A + e^{-t^2}}$$

If we determined A e.g.: $y(0) = 0$, we can plot the graph:



Can we understand the nature of the family of solutions without solving the equation? In general, consider equations of form

$$\frac{dy}{dt} = f(y, t)$$

We consider derivatives for different t (and curves of $\frac{df}{dt} = 0$, called *isoclines*) to obtain vector field/solution curves. The curves cannot cross if $f(y, t)$ is single valued.

4.6.1 Stability of Fixed Points

When y is close to 1, we will be pushed towards 1, but when y is close to -1, we will be pushed away from -1. Mathematically, for a point $y = x$ that makes $\frac{dy}{dt} = 0$ for all t, if $\frac{dy}{dt} < 0$ when $y > x$ and $\frac{dy}{dt} > 0$ when $y < x$, then x is stable. If the signs of the derivative are reversed, x is unstable.

4.6.2 Perturbation Analysis

Suppose that $y = a$is a fixed point, then consider point $y = a + \epsilon(t)$ Then

$$\begin{aligned}\frac{d\epsilon}{dt} &= f(a + \epsilon, t) \\ &= f(a, t) + \epsilon \frac{\partial f}{\partial y}(a, t) + O(\epsilon^2) \\ &= \epsilon \frac{\partial f}{\partial y}(a, t) + O(\epsilon^2) \\ &= [\frac{\partial f}{\partial y}]\epsilon\end{aligned}$$

4.6.3 Autonomous systems

This name represents equations of the form:

$$\frac{dy}{dt} = f(y)$$

Then near a fixed point $y = a$, where $f(a) = 0$, go through the same procedure and reach:

$$\dot{\epsilon} = [\frac{df}{dy}(a)]\epsilon = k\epsilon$$

$$\Rightarrow \epsilon = \epsilon_0 e^{kt}$$

Example 9 (Chemical Reaction)**.**

$$NaOH + HCl \to H_2O + NaCl$$

Initially the concentrations are: $[NaOH] = a_0$, $[HCl] = b_0$, $[H_2O] = 0$, $[NaCl] = 0$

If reactants are in dilute solution (in water say) then reaction rate is proportional to ab, or in equation form:

$$\frac{dc}{dt} = \lambda ab \text{ for some } \lambda$$

$$\frac{dc}{dt} = \lambda(a_0 - c)(b_0 - c) = f(c)$$

It is clear that $c = a_0$ is a stable equilibrium but $c = b_0$ is an unstable one. (Parabola graph)

4.7 Logistic Equation

We have a population size y, a birth rate αy, death rate βy, we get:

$$\frac{dy}{dt} = (\alpha - \beta)y \Rightarrow y = y_0 e^{(\alpha-\beta)y}$$

Population increases or decreases exponentially depending on whether birth rate exceeds death rate or vice versa.

4.7.1 Fighting for limited resources

Probability of finding a piece of food $\propto y$. Probability of finding same piece of food is $\propto y^2$. If food is scarce then they "fight" (to the death), so death rate due to fighting (competing) is γy^2.

$$\frac{dy}{dt} = (\alpha - \beta)y$$

$$= ry(1 - \frac{y}{Y}) \text{ where } r = \alpha - \beta, \text{ and } Y = \frac{r}{\gamma}$$

0 is an unstable fixed point and y is a stable fixed point. When y is small, $y << Y$, then

$$\dot{y} \simeq ry$$

So population grows exponentially but eventually equilibrium y=Y is reached.

4.8 Discrete equations(Difference Equation)

Evolution of species may occur discretely (e.g. births in spring, deaths more common in winter) rather than continuously, so a better model might be of the form

$$x_{n+1} = \lambda x_n(1 - x_n)$$
$$\frac{dy}{dt} = ry(1 - \frac{y}{Y})$$

Approximate the left hand side to give

$$\frac{y_{n+1} - y_n}{\Delta t} \simeq y_n(1 - \frac{y_n}{Y})$$

$$\Rightarrow y_{n+1} \simeq y_n + r\Delta t y_n(1 - \frac{y_n}{Y})$$
$$= (1 + r\Delta t)y_n(1 - (\frac{r\Delta t}{1 + r\Delta t})\frac{y_0}{Y})$$

$$x_{n+1} = \lambda x_n(1 - x_n), \lambda = 1 + r\Delta t,\ x_n = (\frac{r\Delta t}{1 + r\Delta t})\frac{y_0}{Y}$$

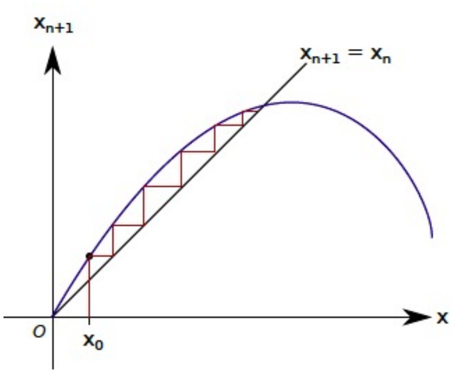
This is the discrete logistic equation or logistic map. It is of the general form $x_{n+1} = f(x_n)$

4.8.1 Behaviour of Logistic map

Fixed points: $x_{n+1} = x_n \Rightarrow f(x_n) = x_n$ For logistic map $\lambda x_n(1 - x_n) = x_n$

$$\Rightarrow x_n = 0 \text{ or } x_n = 1 - \frac{1}{\lambda}$$

For $\lambda < 1$ population dies out.



For $1 < \lambda < 2$, we have a stable fixed point as shown on the left. For $2 < \lambda < 3$, we have the same converging sequence, but it is oscillatory (the sequence forms a decaying ”box” around the stable point). For $3 < \lambda < 1+\sqrt{6}$, we have a 2-cycle (oscillating between two stable points).

5 Second-order Differential Equations

5.1 Constant Coefficients

Definition. A second order linear ordinary differential equation with constant coefficients can be expressed as followed:

$$ay'' + by' + cy = f(x)$$

(i) Find the complementary function satisfying the homogeneous equation $ay'' + by' + cy = 0$

(ii) Find a particular Integral that satisfies the full equation.

5.1.1 Complementary function

$e^{\lambda x}$ is an eigenfunction of $\frac{d}{dx}$ and $\frac{d^2}{dx^2}(= \frac{d}{dx}(\frac{d}{dx}))$. Complementary functions have the form:

$$y_c = e^{\lambda x} \Rightarrow y_c' = \lambda e^{\lambda x} \Rightarrow y_c'' = \lambda^2 e^{\lambda x}$$

$$\Rightarrow a\lambda^2 + b\lambda + c = 0$$

This is the *characteristic equation.* There are two solutions, giving itwo complementary functions

$$y_1 = e^{\lambda_1 x},\ y_2 = e^{\lambda_2 x}$$

if λ_1, λ_2 are distinct. Then y_1, y_2 are linearly independent and complete-they form a basis for the solution space. The general complementary function is

$$y_c = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

Example 10.

$$y'' - 5y' + 6y = 0$$

$$\Rightarrow \lambda_1 = 2, \lambda_2 = 3 \Rightarrow y_c = Ae^{2x} + Be^{3x}$$

5.1.2 Detuning

Example 11.

$$y'' - 4y' + 4 = 0$$

We only have one value of λ. So we need to *detune.* Consider

$$y'' - 4y' + (4 - \epsilon^2)y = 0 \Rightarrow y_c = e^{2x}[Ae^{\epsilon x} + Be^{-\epsilon x}]$$

$$y_c = e^{2x}[A + B + \epsilon x(A - B) + O(a\epsilon^2, b\epsilon^2)]$$

Choose $A + B = \alpha$, $\epsilon(A - B) = \beta$

$$\Rightarrow A = \frac{1}{2}(\alpha + \frac{\beta}{\epsilon}),\ B = \frac{1}{2}(\alpha - \frac{\beta}{\epsilon})$$

Choose that α, β independent of ϵ as epsilon tends to 0.

$$\Rightarrow y_c = e^{2x}[\alpha + \beta x + O\epsilon] \Rightarrow e^{2x}(\alpha + \beta x) \text{ as } \epsilon \text{ tends to } 0$$

This is a demonstration of a general rule that if $y_1(x)$ is a degenerate linear differential equation with constant coefficients then $y_2(x) = xy_1(x)$ is an independent complementary function.

5.1.3 Method for Second Complementary Function

This is associated with a degenerate solution of the homogeneous equation. We can look for a solution in the form

$$y_2(x) = v(x)y_1(x)$$

Example 12.

$$y'' - 4y' + 4y = 0$$

$$y_1 = e^{2x}, \text{Try } y_2 = ve^{2x}$$

$$\Rightarrow y_2' = (v' + 2v)e^{2x}, y_2'' = (v'' + 4v' + 4v)e^{2x}$$

Then solve for v to get the same solution.

5.1.4 Phase Space

A differential equation of nth order

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_o(x)y = f(x)$$

determines the nth derivative $y^{(n)}(x)$ and hence all higher derivatives,in terms of $y(x)$, $y'(x)$,..., $y^{(n-1)}(x)$. Therefore we know the Taylor series that we can use to extend y to values around x.

Definition. Define a solution vector

$$\mathbf{Y}(x) = (y(x), y'(x), \cdots, y^{(n-1)}(x))$$

for each value of x, As x varies $\mathbf{Y}(x)$ traces out a trajectory in phase space.

The solution $y_1(x)$ and $y_2(c)$ are independent solutions of the differential equations iff $\mathbf{Y}_1$, $\mathbf{Y}_2$ are linearly independent vectors in phase space. i.e.: iff the *Wronskian*(determinant) $W(x) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \neq 0$

Theorem (Abel's Theorem)**.** Write equation in standard form:

$$y'' + p(x)y' + q(x)y = 0$$

Then either $W \equiv 0\ \forall x$, or $W \neq 0\ \forall x$. If two solutions are independent for some particular value of x then they re independent for all values of x.

Proof. If y_1 and y_2 are both solutions, then

$$y_1'' + py_1' + qy_1 = 0, \quad \Rightarrow y_2y_1'' + py_2y_1' + qy_2y_1 = 0$$
$$y_2'' + py_2' + qy_2 = 0 \quad \Rightarrow y_1y_2'' + py_1y_2' + qy_1y_2 = 0$$

Subtract, we have:

$$y_2y_1'' - y_1y_2'' + p(y_2y_1' - y_1y_2') = 0 \Rightarrow -W' - pW = 0 \Rightarrow W = W_0e^{\int -p(x)}$$

Where W_0 is a constant. Exponential function is never zero, so either $W_0 = 0 \Rightarrow W \equiv 0$ or W is never 0. □

Note. Any linear nth order differential equation is equal to a system of first order equations. It can be shown that for the system $\dot{\mathbf{Y}} + A\mathbf{Y} = 0$, we have:

$$W' + Tr(A)W = 0 \Rightarrow W = W_0e^{-\int Tr(A)dx} \Rightarrow \text{ Abel's Theorem holds}$$

5.2 Particular Integrals

5.2.1 Guessing

If the forcing terms are easy, we can easily "guess" the particular integral.

$f(x)$	$y_p(x)$
e^{mx}	Ae^{mx}
$\sin kx$ $\cos kx$	$A\sin kx + B\cos kx$
polynomial $p_n(x)$	$q_n(x) = a_nx^n + \cdots + a_1x + a_0$

Remember that the equation is linear! (Consider each forcing term separately).

5.2.2 Detuning

Same as before, if resonance occurs, try $y_p = At\cos x$ to get an oscillator vibrating at tis natural frequency. Alternatively, for forcing term $\sin\omega_0 t$, one could try $y_p = C(\sin\omega t - \sin\omega_0 t)$ and then take $\omega - \omega_0$ limit to 0. (Recommend former way on test)

5.2.3 Variation of parameters

Let $y_1(x)$ and $y_2(x)$ be linearly independent complementary functions of the ODE $y'' + p(x)y' + q(x)y = f(x)$. The solution can be written in terms of $\mathbf{Y}_1 = (y_1, y_1')$, $\mathbf{Y}_2 = (y_2, y_2')$:

$$y_p = uy_1 + vy_2 \tag{a}$$

$$y_p' = uy_1' + vy_2' \tag{b}$$

From the second equation, we have

$$y_p'' = (uy_1'' + u'y_1') + (vy_2'' + v'y_2') \tag{c}$$

If we use consider (c) + p(b) + q(a), we have $y_1'u' + y_2'v' = f$.

By (a)$'$ - (b), we obtain $y_1u' + y_2v' = 0$. Now we have two simultaneous equations for u' and v'.

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{W} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix}$$

So $u' = -\frac{y_2}{W}f$ and $v' = \frac{y_1}{W}f$.

Example 13.

$$y'' + 4y = \sin 2x \Rightarrow y_1 = \cos 2x,\ y_2 = \sin 2x \Rightarrow y_p = 2u\cos 2x - 2v\sin 2x$$

$$\Rightarrow u = \frac{-\cos 4x}{16},\ v = \frac{\sin 4x}{16} - \frac{x}{4}$$

$$\Rightarrow y_p = \frac{1}{16}(-\cos 4x\sin 2x + \sin 4x\cos 2x) - \frac{x}{4}\cos 2x = \frac{1}{16}\sin 2x - \frac{1}{4}x\cos 2x$$

5.3 Homogeneous equations, a.k.a Linear equidimensional equations

Definition. This class of equations look like:

$$ax^2y'' + bxy' + cy = f(x)$$

5.3.1 Solving

(i) Use eigenfunction $y = x^k$ to reach characteristic equation $ak(k-1) + bk + c = 0$. Then find particular solution.

(ii) Use substitution $z = \ln x$ to transform this equation into $a^2\frac{d^2y}{dz^2} + (b-a)\frac{dy}{dz} + cy = f(e^z)$. This method agrees with the one above.

(iii) For degenerate solutions, use $x^k\ln x$. Same with resonant forcing.

5.4 Difference Equation

Definition. This class of equations is defined as:

$$ay_{n+2} + by_{n+1} + cy_n = f(n)$$

Solve in a similar way to differential equation. We can think of a difference operator$D[y_n] = y_{n+1}$ has eigenfunction $y_n = k^n$ General complementary function has characteristic equation:

$$ak^2 + bk + c = 0$$

So the solutions are of form:

$$a_n^{(c)} = Ak_1^n + Bk_2^n \text{ if } k_1 \neq k_2$$
$$a_n^{(c)} = (A + Bn)k^n \text{ if } k_1 = k_2$$

5.4.1 Guesswork

f_n	y_n^p
k^n	Ak^n
k^n	nk^n (if degeneracy occurs)
n^p	$A_1 n^p + A_2 n^{p-1} + \cdots + A_p$

5.5 Transcients and damping

For physical systems, often there is a restoring force (suspension).

Consider a car of mass M with a vertical force $F(t)$ acting on it . We can consider the wheels to be springs ($F = kx$) with a "shock absorber" ($F = l\dot{x}$).

$$M\ddot{x} = F(t) - kx - l\dot{x}.$$

So we have

$$\ddot{x} + \frac{l}{M}\dot{x} + \frac{k}{M}x = F(t).$$

Write $t = \tau\sqrt{M/k}$, where τ is dimensionless. Then we obtain

$$\ddot{x} + 2\kappa\dot{x} + x = f(\tau)$$

where, $\dot{x}$ means $\frac{\mathrm{d}x}{d\tau}$, $\kappa = \frac{l}{2\sqrt{kM}}$ and $f = \frac{F}{k}$.

5.5.1 Free (natural) response $f = 0$

$$\ddot{x} + 2\kappa\dot{x} + x = 0$$

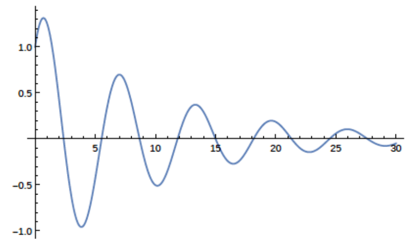
We try $x = e^{\lambda\tau}$

$$\lambda + 2\kappa\lambda + 1 = 0$$
$$\lambda = -\kappa \pm \sqrt{\kappa^2 - 1}$$
$$= \lambda_1, \lambda_2$$

5.5.2 Underdamped

If $\kappa < 1$, we have $x = e^{-\kappa\tau}(A \sin\sqrt{1-\kappa^2}\tau + B\cos\sqrt{1-\kappa^2}\tau)$.

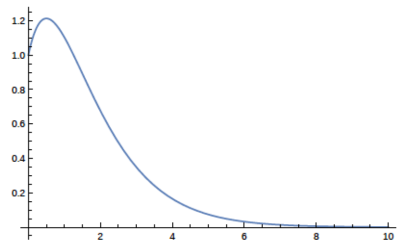
The period is $\frac{2\pi}{\sqrt{1-\kappa^2}}$ and its amplitude decays in a characteristic of $O(\frac{1}{\kappa})$. Note that the damping increases the period. As $\kappa \to 1$, then the oscillation period $\to \infty$.



5.5.3 Critically damped

If $\kappa = 1$, then $x = (A + B\tau)e^{-\kappa\tau}$.

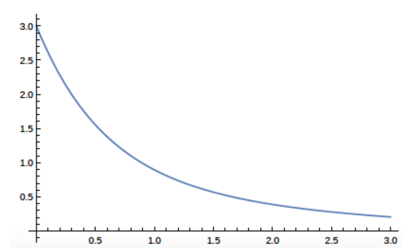
The rise time and decay time are both $O(\frac{1}{\kappa}) = O(1)$. So the dimensional rise and decay times are $O(\sqrt{M/k})$.



5.5.4 Overdamped

If $\kappa > 1$, then $x = Ae^{-\lambda_1\tau} + Be^{-\lambda_2\tau}$ with $\lambda_1 < \lambda_2$. Then the decay time is $O(1/\lambda_1)$ and the rise time is $O(1/\lambda_2)$.

Note that it is possible to get a large initial increase in amplitude.



5.5.5 Forcing

In a forced system, the CF typically determine the transcient response, while the PI determines the asymptotic response.

For example, if $f(\tau) = \sin\tau$, $x_p = C\sin\tau + D\cos\tau$. In this case, $x_p = -\frac{1}{2\kappa}\cos\tau$. The general solution is $x = Ae^{-\lambda_1\tau} + Be^{-\lambda_2\tau} - \frac{1}{2\kappa}\cos\tau \sim -\frac{1}{2\kappa}\cos\tau$ as $\lambda \to \infty$ since $\mathbf{Re}(\lambda_{1,2}) > 0$.

Note. The forcing response is out of phase with the forcing.

5.6 Impulses and Point Forces

We don't need to know the details of $F(t)$ (bouncing ball) but only note that it acts for a short time $O(\epsilon)$. To make it convenient for mathematics, we take limit $\epsilon \to 0$. Newton's 2nd law gives:

$$m\ddot{x} = F(t) - mg$$

Integrating:

$$\int_{T-\epsilon}^{T+\epsilon} \frac{d^2x}{dt^2} dt = \int_{T-\epsilon}^{T+\epsilon} F(t)dt - \int_{T-\epsilon}^{T+\epsilon} mg \text{ dt} \Rightarrow [m\frac{dx}{dt}]_{T-\epsilon}^{T+\epsilon} = I - 2\epsilon mg$$

Gravity ignored, Impulse written as I we have:

$$I = [m\frac{dx}{dt}]_{T-\epsilon}^{T+\epsilon}$$

we only need the integral properties of $F(t;\epsilon)$. Consider a family of function $D(t,\epsilon)$ such that

$$\lim_{\epsilon\to 0} = 0 \text{ for all } t \neq 0 \ \& \ \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} D(t;\epsilon)dt = 1$$

We define the Dirac Delta function by:

$$\delta(x) = \lim_{\epsilon\to 0} D(x:\epsilon)$$

On the understanding that we can only use its integral properties. (its value at 0 is undefined) For example:

$$\int_{-\infty}^{\infty} g(x)\delta(x)dx = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} g(x)D(x;\epsilon)dx = g(0)\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} D(x;\epsilon)dx = g(0)$$

In general, the integral is only non-zero if the "infinite" point is between the limits of the integral, and the value is the value of $g(x)$ at that point. If the limit is at the infinite point, it is undefined.

5.7 Heaviside Step Function

Definition.

$$H(x) = \int_{-\infty}^{x} \delta(t)dt$$

$$H(x) = 0 \text{ for } x < 0\ , H(x) = 1 \text{ for } x > 0, H(x) \text{ undefined for } x = 0$$

5.7.1 Solving differential equations with discontinuous functions

Separate the equation into parts before discontinuity and after discontinuity and then consider boundary conditions (continuity of function).

6 Series Solution

Consider the following type of equations:

$$p(x)y''(x) + q(x)y'(x) + r(x)y = 0$$

Definition. A point $x = x_0$ is an *ordinary point* if $\frac{Q}{P}$ and $\frac{R}{P}$ have taylor series (are analytic) about x_0.

If it is not an ordinary point, It is an *regular singular point* if the two fractions can have taylor series when it is expressed in form:

$$p(x)(x-x_0)^2y''(x) + q(x)(x-x_0)y'(x) + r(x)y = 0$$

If these two fractions also does not have taylor series. It is an *irregular singular point.*

Example 14.

$$(1-x^2)y'' - 2xy' + 2y = 0$$

$x = 0$ is an ordinary point, $x = \pm 1$ are ordinary singular points.

Proposition 1. *If x_0 is an ordinary point, then equation has two linearly independent solutions of the form*

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

convergent in some range about x_0. If x_0 is regular singular , then equation has ≥ 1 solution in:

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\sigma} = (x - x_0)^{\sigma} f(x) \ , \ a_0 \neq 0$$

This is called a Frobenius series*. $f(x)$ is analytic, σ can be any complex number.*

Example 15.

$$(1-x^2)y'' - 2xy'' + 2y = 0$$

Try solution

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Now write it in equidimensional form:

$$(1-x^2)x^2y'' - 2x^2xy' + 2x^2y = 0$$

$$\sum a_n((1-x^2)n(n-1) - 2x^2n + 2x^2)x^n = 0$$

Coefficient of x^n gives the general recurrence relation:

$$n(n-1)a_n + (-(n-2)(n-3) - 2(n-2) + 2)a_{n-2} = 0$$

$$\Rightarrow n(n-1)a_n = (n^2 - 3n)a_{n-2}$$

$$n = 0, n = 1: \ 0a_0 = 0, 0a_1 = 0 \Rightarrow \ a_0, \ a_1 \text{ are arbitrary.}$$

$$n > 1: \ a_n = \frac{n-3}{n-1}a_{n-2} = \frac{n-5}{n-1}a_{n-4} = \cdots$$

$$a_{2k} = \frac{-1}{2k-1}a_0, a_{2k+1} = 0$$

$$y = a_0[1 - \frac{x^2}{1} - \frac{x^4}{3} - \frac{x^6}{5} - \cdots] + a_1x = a_0[1 - \frac{x}{2}\ln(\frac{1+x}{1-x})] + a_1x$$

Example 16 (Frobenius Series)**.**

$$4xy'' + 2(1-x^2)y' - xy = 0$$

$$4x^2y'' + 2(1-x^2)(xy') + x^2y = 0$$

Now $x=0$ is a regular singular point, try Frobenius series:

$$[4(n+\sigma)(n+\sigma-1)+2(1-x^2)(n+\sigma)-x^2]a_n x^{n+\sigma}=0$$

$$[4(n+\sigma)(n+\sigma-1)+2(n+\sigma)]a_n+[-2(n+\sigma-2)-1]a_{n-2}=0$$

$$2(n+\sigma)(2n+2\sigma-1)a_n=(2n+2\sigma-3)a_{n-2}$$

$n=0$ gives the indicial equation for the index σ.

$$2\sigma(2\sigma-1)a_0=0\Rightarrow\sigma=0,\frac{1}{2}$$

If $\sigma=0$

$$2n(2n-1)a_n=(2n-3)a_{n-2}$$

For $n=0$, we can see a_0 is arbitrary. For $n=1$,we can see $a_1=a_3=a_5\cdots=0$

$$y=a_0[1+\frac{1}{4\cdot 3}x^2+\frac{5}{8\cdot 7\cdot 4\cdot 3}x^4+\cdots]$$

$\sigma=\frac{1}{2}$ we have:

$$(2n+1)2na_n=(2n-2)a_{n-2}$$

$n=0$ shows that a_0 is arbitrary, and for $n=1$, we don't have odd terms (again):

$$\Rightarrow y=x^{\frac{1}{2}}b_0[1+\frac{1}{2\cdot 5}x^2+\frac{3}{2\cdot 5\cdot 4\cdot 9}x^4+\cdots]$$

6.1 Behaviour near $x=x_0$

Indicia equation has two roots σ_1 and σ_2. If $\sigma_2-\sigma_1$ is not an integer then there are two linearly independent Frobenius solutions

$$y=(x-x_0)^{\sigma_1}\sum_{n=0}^{\infty}a_n(x-x_0)^n+(x-x_0)^{\sigma_2}\sum_{n=0}^{\infty}b_n(x-x_0)^n$$

If $\sigma_2-\sigma_1$ is an integer then there is a Frobenius solution of the form:

$$y_1=(x-x_0)^{\sigma_2}\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

where $\sigma_2\geq\sigma_1$

The other solution is (usually) of form

$$y_2=\ln(x-x_0)y_1+\sum_{n=0}^{\infty}b_n(x-x_0)^n$$

7 Directional Derivatives

Consider a displacement $d\mathbf{s} = (\mathrm{d}x, \mathrm{d}y)$ of function $f(x, y)$. The change during that displacement is:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\Rightarrow df = (dx, dy) \cdot (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = d\mathbf{s} \cdot \nabla f$$

$(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ are components of gradient of f, denoted *grad* f or ∇f. Write $d\mathbf{s} = \hat{\mathbf{s}}\ ds$ where $|\hat{\mathbf{s}}| = 1$.Then:

$$\frac{df}{ds} = \hat{\mathbf{s}} \cdot \nabla f$$

where $\frac{df}{ds}$ is the directional derivative of f. This is the definition of ∇f.

7.0.1 The gradient vector of $f(x, y)$

$$\frac{df}{ds} = |\nabla f| \cos\theta$$

where θ is the angle between ∇f and $d\mathbf{s}$. Therefore, maximum occurs when $d\mathbf{s}$ points along ∇f.

(i) ∇f has magnitude of the maximum rate of change over all directions (in $x - y$ plane) at a point. It has direction in which the graph increases most rapidly.

(ii) If $d\mathbf{s}$ is a displacement along a contour of f then ∇f is orthogonal to the contour of f.

$$\frac{df}{ds} = 0 \Rightarrow \hat{\mathbf{s}} \cdot \nabla f = 0$$

7.1 Stationary Points

There is always one direction in which $\frac{df}{ds} = 0$, namely parallel to a contour of f. Local maxima and minima have

$$\frac{df}{ds} = \hat{\mathbf{s}} \cdot \nabla f = 0$$

for *all* directions. so $\nabla f = 0$.

(i) For maximum/minimum points, ∇f points toward/away the point.

(ii) For saddle points, contours cross (it *only* crosses for saddle points).

7.2 Taylor series for Multivariable Functions

Consider finite displacement $\delta\mathbf{s}$ along a straight line in $x - y$ plane. Then

$$\delta\mathbf{s}\frac{d}{ds} \equiv \delta\mathbf{s} \cdot \nabla$$

So the Taylor series along the line is:

$$f(s) = f(s_0 + \delta s) = f(s_0) + \delta s\frac{df}{ds} + \frac{1}{2}(\delta s)^2\frac{d^2 f}{ds^2} + \cdots = f(s_0) + \delta\mathbf{s} \cdot \nabla f + \frac{1}{2}(\delta s)^2(\hat{\mathbf{s}} \cdot \nabla)(\hat{\mathbf{s}} \cdot \nabla)f + \cdots$$

Now:

$$\delta\mathbf{s}\cdot\nabla f = \delta x\frac{\partial f}{\partial x} + \delta y\frac{\partial f}{\partial y}$$

Expanding the terms:

$$\begin{aligned}(\delta s)^2(\hat{\mathbf{s}}\cdot\nabla)(\hat{\mathbf{s}}\cdot\nabla)f &= (\delta s)^2(\hat{\mathbf{s}}_x\frac{\partial}{\partial x} + \hat{\mathbf{s}}_y\frac{\partial}{\partial y})(\hat{\mathbf{s}}_x\frac{\partial f}{\partial x} + \hat{\mathbf{s}}_y\frac{\partial f}{\partial y})\\ &= \delta x^2\frac{\partial^2 f}{\partial x^2} + \delta x\delta y\frac{\partial^2 f}{\partial x\partial y} + \delta y\delta x\frac{\partial^2 f}{\partial y\partial x} + \delta y^2\frac{\partial^2 f}{\partial y^2}\\ &= \begin{pmatrix}\delta x & \delta y\end{pmatrix}\begin{pmatrix}f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}\end{aligned}$$

where

$$\begin{pmatrix}f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{pmatrix} \equiv \nabla\nabla f \text{ (Hessian matrix)}$$

In general, the coordinate-free form of Taylor expansion is is

$$f(\mathbf{x}) = f(\mathbf{x_0}) + \delta\mathbf{x}\cdot\nabla f(\mathbf{x}_0) + \frac{1}{2}\delta\mathbf{x}\cdot\nabla\nabla f\cdot\delta\mathbf{x}$$

7.3 Classification of stationary points

At a stationary point $\mathbf{x}_0$, we know that $\nabla f(\mathbf{x}_0) = 0$. So at a point near the stationary point,

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \frac{1}{2}\delta\mathbf{x}\cdot H\cdot\delta\mathbf{x},$$

where $H = \nabla\nabla f(\mathbf{x}_0)$ is the Hessian matrix.

At a minimum, $\delta\mathbf{x}\cdot H\cdot\delta\mathbf{x} > 0$ for all $\delta\mathbf{x}$, so it is *positive definite*. Similarly, at a maximum, $\delta\mathbf{x}\cdot H\cdot\delta\mathbf{x} < 0$ for all $\delta\mathbf{x}$ so it is negative definite. At a saddle, $\delta\mathbf{x}\cdot H\cdot\delta\mathbf{x}$ is indefinite. If $\det H = 0$, then we need to look at higher derivatives.

Now note that $H = \nabla\nabla f$ is symmetric ($f_{xy} = f_{yx}$). So H is diagonalizable. W.r.t these axes in which H is diagonal (principal axes), we have

$$\delta\mathbf{x}\cdot H\cdot\delta\mathbf{x} = \lambda_1(\delta x)^2 + \lambda_2(\delta y)^2 + \cdots + \lambda_n(\delta z)^2$$

where $\lambda_1, \lambda_2, \cdots \lambda_n$ are the eigenvalues of H.

So positive-definite means $\lambda_i > 0$ for all i. Similarly, it is negative-definite iff $\lambda_i < 0$ for all i. If eigenvalues have mixed sign, then it is a saddle point.

7.3.1 Determination of definiteness

Definition (Signature of Hessian matrix)**.** The *signature* of H is the pattern of the signs of the subdeterminants:

$$\underbrace{f_{xx}}_{|H_1|}, \underbrace{\begin{vmatrix}f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{vmatrix}}_{|H_2|}, \cdots, \underbrace{\begin{vmatrix}f_{xx} & f_{xy} & \cdots & f_{xz}\\ f_{yx} & f_{yy} & \cdots & f_{yz}\\ \vdots & \vdots & \ddots & \vdots\\ f_{zx} & f_{zy} & \cdots & f_{zz}\end{vmatrix}}_{|H_n|=|H|}$$

Proposition 2. *H is positive definite if and only if the signature is $+, +, \cdots, +$. H is negative definite if and only if the signature is $-, +, \cdots, (-1)^n$. Otherwise, H is indefinite.*

7.3.2 Contours of $f(x, y)$

Consider H in 2 dimensions, and diagonal axes for H. So $H = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Write $\mathbf{x} - \mathbf{x}_0 = (X, Y)$.

Then near $\mathbf{x}_0$, f = constant $\Rightarrow \mathbf{x}H\mathbf{x}$ = constant, i.e. $\lambda_1 X^2 + \lambda_2 Y^2$ = constant. At a maximum or minimum, λ_1 and λ_2 have the same sign. So these countours are locally ellipses. At a saddle point, they have different signs and the contours are locally hyperbolae.

8 Systems of Linear differential equations

Consider two dependent variables $y_1(t)$ and $y_2(t)$ with

$$\dot{y_1} = ay_1 + by_2 + f_1(t)$$

$$\dot{y_2} = cy_1 + dy_2 + f_2(t)$$

$$\Rightarrow \dot{\mathbf{Y}} = M\mathbf{Y} + \mathbf{F}$$

where $\mathbf{Y}$=$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ M=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{F}$= $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

This can be simplified to a second-order linear ODE of form below through differentiation:

$$\ddot{y_1} + A\dot{y_1} + Ay_1 = f(t)$$

Transferring back is equally easy:

$$\ddot{y} + a\dot{y} + by = f, \text{ let } y_1 = y, y_2 = \dot{y}$$

To transfer this back into a system of two linear first order differential equations.

Complementary solution $\mathbf{Y}_c = \mathbf{v}e^{\lambda t}$ has to satisfy:

$$M\mathbf{v} = \lambda\mathbf{v}$$

So $\mathbf{v}$ is an eigenvector of M with corresponding eigenvalue λ. Thus the solution (non-degenerate) is:

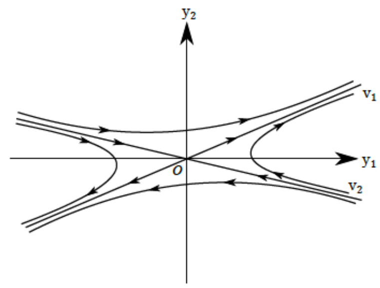
$$\mathbf{Y} = A\mathbf{v}_1 e^{\lambda_1 x} + B\mathbf{v}_1 e^{\lambda_1 x} + f(x)$$

where $\mathbf{v}_1, \mathbf{v}_2$ are the eigenvectors, λ_1, λ_2 are the eigenvalues, and $f(x)$ the forcing term.

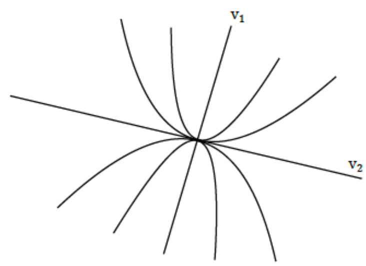
8.1 Phase-space trajectories

There are three possible cases of solutions corresponding to three different possible eigenvalues of M:

(i) If both λ_1, λ_2 are real with opposite sign ($\lambda_1\lambda_2 < 0$). wlog assume $\lambda_1 > 0$. Then there is a saddle as above:

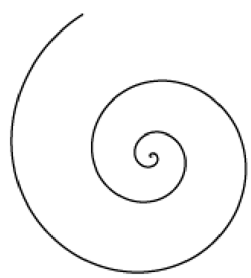


(ii) If λ_1, λ_2 are real with $\lambda_1 \lambda_2 > 0$. wlog assume $|\lambda_1| \geq |\lambda_2|$. Then the phase portrait is



If both $\lambda_1, \lambda_2 < 0$, then the arrows point towards the intersection and we say there is a stable node. If both are positive, they point outwards and there is an unstable node.

(iii) If λ_1, λ_2 are complex conjugates, then we obtain a spiral



If $\mathbf{Re}(\lambda_{1,2}) < 0$, then it spirals inwards. If > 0, then it spirals outwards. If $= 0$, then we have ellipses with common center. We can determine whether the spiral is positive (as shown above), or negative (mirror image of the spiral above) by considering the eigenvectors.

8.2 Nonlinear dynamical systems

Consider the second-order autonomous system (i.e. t does not explicitly appear in the forcing terms on the right)

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

It can be difficult to solve the equations, but we can learn a lot about phase-space trajectories of these solutions by studying the equilibria and their stability.

8.2.1 Equilibrium (fixed) points

Definition (Equilibrium point)**.** An *equilibrium* point is a point in which $\dot{x} = \dot{y} = 0$ at $\mathbf{x}_0 = (x_0, y_0)$.

Clearly this occurs when $f(x_0, y_0) = g(x_0, y_0) = 0$. We solve these simultaneously for x_0, y_0.

8.2.2 Stability

Write $x = x_0 + \xi$, $y = y_0 + \eta$. Then

$$\begin{aligned}\dot{\xi} &= f(x_0 + \xi, y_0 + \eta)\\ &= f(x_0, y_0) + \xi\frac{\partial f}{\partial x}(\mathbf{x}_0) + \eta\frac{\partial f}{\partial y}(\mathbf{x}_0) + O(\xi^2, \eta^2)\end{aligned}$$

So if $\xi, \eta \ll 1$,

$$\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\end{pmatrix} = \begin{pmatrix}f_x & f_y\\ g_x & g_y\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

This is a linear system, and we can determine its character from the eigensolutions.

9 Partial differential equations (PDEs)

9.1 First-order wave equation

Consider the equation of the form

$$\frac{\partial y}{\partial t} = c\frac{\partial y}{\partial x}$$

with c constant. Recall that along a path $x = x(t)$ so that $y = y(x(t), t)$,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial y}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial y}{\partial t}$$

by the chain rule. Now we choose a path along which

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -c. \tag{1}$$

Along such paths,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -c\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = 0 \tag{2}$$

So wex replaced the original partial differential equations with a pair of ordinary differential equations.

(2) tells us y is constant along the path. (1) says $x = x_0 - ct$, with x_0 constant. Write $x + ct = x_0$.

So we have a family of paths, each determined by x_0, and along each path, y is constant. For each value of x_0, we obtain a unique constant function along the path $y = f(x_0)$. So

$$y = f(x + ct),$$

which is the general solution of y.

As we move up the time axis, we simply take the $t = 0$ solution and translate it to the left.

The paths we've identified are called the "characteristics" of the wave equation. (In this example it is constant). We can also solve forced equations, such as

$$\frac{\partial y}{\partial t} + 5\frac{\partial y}{\partial x} = e^{-t}, \quad y(x, 0) = e^{-x^2}.$$

So we obtain $\frac{\mathrm{d}y}{\mathrm{d}t} = e^{-t}$ with $\frac{dx}{\mathrm{d}t} = -5$. So $y = A - e^{-t}$ on paths $x = x_0 + 5t$.

At $t = 0$, $y = A - 1$ and $x = x_0$. So $A - 1 = e^{-x_0^2}$ from the given initial conditions., i.e. $A = 1 + e^{-x_0^2}$. So

$$y = 1 + e^{-(x-5t)^2} - e^{-t}.$$

9.2 Second-order wave equation

We consider equations in the following form:

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

Suppose that $\rho(x)$ is the mass per unit length of a string. Then the force $ma = \rho\frac{\partial^2 y}{\partial t^2}$ is proportional to the second derivative $\frac{\partial^2 y}{\partial x^2}$ (curvature of string)

This is often known as the "hyperbolic equation" because the form resembles that of a hyperbola (which has the form $x^2 - b^2y^2 = 0$). But the connection stops there. If c is constant, we write

$$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = 0 \Rightarrow (\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})y = 0$$

If $y = f(x+ct)$, then the first operator differentiates it to give a constant (as in the first-order wave equation). Then applying the second operator differentiates it to 0. So $y = f(x + ct)$ is a solution.

Since the operators are commutative, $y = f(x - ct)$ is also a solution. So the general solution is

$$y = f(x + ct) + g(x - ct).$$

This shows that the solution composes of superpositions of waves travelling to the left and those to the right.

We can show that this is indeed the most general solution by substituting $\xi = x + ct$ and $\eta = x - ct$. We can show, using the chain rule, that $y_{tt} - c^2y_{xx} \equiv -4c^2y_{\eta\xi} = 0$. Integrating twice gives $y = f(\xi) + g(\eta)$.

How many boundary conditions do we need to have a unique solution? In PDEs, we have to count all variables. In this case, we need 2 boundary conditions and 2 initial conditions.

Example 17.

$$y = \frac{1}{1 + x^2}, \frac{\partial y}{\partial t} = 0 \text{ at } t = 0$$
$$y \to 0 \text{ as } x \to \pm\infty$$

Solution has the form:

$$y = f(x + ct) + g(x - ct)$$

Solving:

$$y = f(x) + g(x) = \frac{1}{1 + x^2}$$

$$\frac{\partial y}{\partial t} = cf'(x) - cg'(x) = 0$$

Solving we have $f' = g'$ and from boundary condition $f = g$ so solution is:

$$f(x) = g(x) = \frac{\frac{1}{2}}{1 + x^2}$$

So:

$$y = \frac{1}{2}[\frac{1}{1 + (x + ct)^2} + \frac{1}{1 + (x - ct)^2}]$$

10 The Diffusion Equation

Definition. Heat conduction in a solid in one dimension is modelled by the diffusion equation:

$$\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2}$$

This is a parabolic PDE where T(x,t) is temperature and the constant k is diffusivity.

Example 18. Infinitely long bar heated at one end. Note in general that "velocity" is proportional to curvature.

There is a similarity solution of the diffusion equation, valid on an infinite domain in which

$$T(x, t) = \theta(\eta)$$

where $\eta = \frac{x}{2\sqrt{\kappa t}}$ So solving for the derivatives:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{4\kappa t}\theta''(\eta)$$

$$\frac{\partial T}{\partial t} = -\frac{\eta}{2t}\theta'$$

In diffusion equation:

$$-\eta\theta' = \kappa\frac{1}{4\kappa}\theta''$$

$$\Rightarrow \theta'' + 2\eta\theta' = 0$$

This is an ordinary differential equation for $\theta(\eta)$. Using integrating factor treating θ' as the variable, we have:

$$\theta' = Ae^{-\eta^2}$$

$$\Rightarrow \theta = A\int_0^\eta e^{-u^2}du + B$$

$$= \alpha \operatorname{erf} \eta + B$$

where:

$$\operatorname{erf} y = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-u^2}du$$

This solution is valid for $\sqrt{\kappa t} << L \Rightarrow t << \frac{L^2}{\kappa}$