Math Tripos Part IA: Dynamics and Relativity

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May 13, 2015

Basic concepts
Space and time, frames of reference, Galilean transformations. Newton's laws. Dimensional analysis. Examples of forces, including gravity, friction and Lorentz. [4]

Newtonian dynamics of a single particle
Equation of motion in Cartesian and plane polar coordinates. Work, conservative forces and potential energy, motion and the shape of the potential energy function; stable equilibria and small oscillations; effect of damping.

Angular velocity, angular momentum, torque.

Orbits: the $u(\theta)$ equation; escape velocity; Kepler's laws; stability of orbits; motion in a repulsive potential (Rutherford scattering). Rotating frames: centrifugal and coriolis forces. *Brief discussion of Foucault pendulum.* [8]

Newtonian dynamics of systems of particles
Momentum, angular momentum, energy. Motion relative to the centre of mass; the two body problem. Variable mass problems; the rocket equation. [2]

Rigid bodies
Moments of inertia, angular momentum and energy of a rigid body. Parallel axes theorem. Simple examples of motion involving both rotation and translation (eg. rolling). [3]

Special relativity
The principle of relativity. Relativity and simultaneity. The invariant interval. Lorentz transformations in $(1 + 1)$-dimensional spacetime. Time dilation and length contraction. The Minkowski metric for $(1 + 1)$-dimensional spacetime. Lorentz transformations in $(3 + 1)$ dimensions. 4-vectors and Lorentz invariants. Proper time. 4-velocity and 4-momentum. Conservation of 4-momentum in particle decay. Collisions. The Newtonian limit. [7]

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1 Newtonian dynamics of particles

Definition. A *particle* is an object of insignificant size. It has *mass* ($m > 0$) and *electric charge* (q). Its position at time t is described by its *position vector*, $\mathbf{r}(t)$ or $\mathbf{x}(t)$, with respect to an origin O. The Cartesian components of $\mathbf{r}$ and $\mathbf{x}$ are measured with respect to orthogonal axes Ox, Oy, and Oz. This choice of origin and axes define a *frame of reference.*

Definition. The *velocity* of the particle is $\mathbf{v} = \mathbf{u} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$ and is tangent to the path or trajectory while its momentum is $\mathbf{p} = m\mathbf{v}$. Its *acceleration* is $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2}$.

1.1 Newton's Laws of Motion

Law.

(i) A body remains at rest, or moves uniformly in a straight line, unless acted on by a force.

(ii) The rate of change of momentum of a body is equal to the force acting on it, a vector equality.

(iii) To every action there is an equal and opposite reaction.

Here a body means either a particle or the centre of mass of an extended object.

1.2 Inertial Frames and the First Law

Newton's laws hold only in special frames of reference that are not accelerating. Such frames are called *inertial frames.* In an inertial frame, $\ddot{\mathbf{r}} = \mathbf{0}$ when no force acts. So Law 1 asserts that inertial frames exist.

1.3 Galilean transformations

Inertial frames are not unique. If S is an inertial frame, then any other frame S' in uniform motion $\mathbf{v}$ relative to S is also an inertial frame. Thus the *Galilean transformation* or a *boost* is:

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t$$

Definition. A general Galilean transformation combines a boost with a translation of space (changing origin time-independently), translation of time (starting clock at different times), rotations (and reflection) of space ($r' = Rr$). These generate the Galilean group.

Law (Galilean relativity)**.** The *principle of relativity* asserts that the laws of physics are the same in inertial frames.

In other words, the equations of Newtonian physics must have *Galilean invariance.*

Since the laws of physics are the same regardless of your velocity, velocity must be a *relative concept*, and there is no such thing as an "absolute velocity" that all inertial frames agree on. But, every inertial frame agrees on whether you are accelerating or not, thus acceleration is *absolute.*

1.4 Newton's Second Law

Law. The *equation of motion* for a particle subject to a force F is

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F},$$

where $\mathbf{p}$ is the (linear) momentum of the particle. We say m is the (inertial) mass of the particle, or its reluctance to accelerate.

$$\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{r}}.$$

if m is constant. If $\mathbf{F}$ is a function of $\mathbf{r}$, $\dot{\mathbf{r}}$ and t, then we have a second-order ordinary differential equation for $\mathbf{r}$.

To determine the solution, we need to specify the initial values of $\mathbf{r}$ and $\dot{\mathbf{r}}$, ie. the initial position and velocity. The trajectory of the particle is then uniquely determined for all future (and past) times.

2 Dimensional Analysis

Physical quantities are not pure numbers, but have *dimensions.* In any equation, the dimensions have to be consistent.

For many problems in dynamics, three basic dimensions are sufficient:

- length, L
- mass, M
- time, T

Physical constants also have dimensions, eg. $[G] = L^3M^{-1}T^{-2}$ by $F = GMm/r^2$.

We can only take sums and products of terms that have dimensions, and if we sum two terms, they must have the same dimension.

2.1 Units

A *unit* is a convenient standard physical quantity. In the SI system, there are base units corresponding to the basics dimensions. The three we need are

- meter (m) for length
- kilogram (kg) for mass
- second (s) for time

A physical quantity can be expressed as a pure number times a unit with the correct dimensions, eg.

$$G = 6.67 * 10^{-11} m^3 kg^{-1} s^{-2}$$

It is important to realize that SI units are chosen arbitrarily and does not affect physics.

2.2 Scaling

Suppose we believe that a physical quantity Y depends on 3 other physical quantities X_1, X_2, X_3, ie. $Y = Y(X_1, X_2, X_3)$. Let their dimensions be as follows:

- $[Y] = L^a M^b T^c$
- $[X_i] = L^{a_i} M^{b_i} T^{c_i}$

Suppose further that we know that the relationship is a power law, ie.

$$Y = C X_1^{p_1} X_2^{p_2} X_3^{p_3},$$

where C is a dimensionless constant (ie. pure number). Since the dimensions must work out, we know that

$$\begin{aligned} a &= p_1 a_1 + p_2 a_2 + p_3 a_3 \\ b &= p_1 b_1 + p_2 b_2 + p_3 b_3 \\ c &= p_1 c_1 + p_2 c_2 + p_3 c_3 \end{aligned}$$

for which there is a unique solution provided that the dimensions of X_1, X_2 and X_3 are independent.

Note that if X_i are not independent, eg. $X_1^2 X_2$ is dimensionless, then our law can involve more complicated terms such as $\exp(X_1^2 X_2)$ since the argument of exp is dimensionless.

If $n > 3$, then the dimensions of $x_1, x_2, \cdots, x_n$ cannot be independent. Order these variables so that $[x_1], [x_2], [x_3]$ are independent and from the remaining variables form $n - 3$ dimensionless groups. For example, for x_4, we have:

$$\lambda = \frac{x_4}{x_1^{q_1} x_2^{q_2} x_3^{q_3}}$$

For some appropriate power. Then the relationship must be of the form:

$$Y = f(\lambda, \mu, \cdots) x_1^{p_1} x_2^{p_2} x_3^{p_3}$$

where f is a dimensionless function of the dimensionless variables. Formally, it is called *Buckingham's Pi Theorem.*

Example. For a simple pendulum having an amplitude of d, weight of m, and the cord having length of l with simple gravity g, we want to find period P.

P could depend on mass ($[m] = M$), length($[l] = L$), initial displacement ($[d] = L$), and gravity($[g] = LT^{-2}$). And obviously $[P] = T$. The initial displacement is dependent on the dimensions of other three, so we form dimensionless constant $\frac{d}{l}$. The relationship must be of the form

$$P = f(\frac{d}{l}) m^{p_1} l^{p_2} g^{p_3}$$

For the dimensions to balance,

$$T = M^{p_1} L^{p_2} L^{p_3} T^{-2p_3} \Rightarrow p_1 = 0, p_2 = -p_3 = \frac{1}{2}$$

$$\Rightarrow P = f(\frac{d}{l}) \sqrt{\frac{l}{g}}$$

The idea of physical similarity, such as all pendulum released from he same angle are similar, is just a rescaled version of the same problem.

3 Forces

3.1 Force and potential energy in one dimension

Consider a particle of mass m moving in a straight line with position $x(t)$. Suppose $F = F(x)$, depends on position but not on velocity or time.

Definition. We define the potential energy $V(x)$ as $F = -\frac{dV}{dx}$. it has an arbitrary constant (so we need to define zero point).

The equation of motion is then

$$m\ddot{x} = -\frac{dV}{dx}$$

We claim the mechanical energy ($KE + PE = \frac{1}{2}m\dot{x}^2 + V(x)$) is conserved in any trajectory satisfying the equation above.

Proof.

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} = \dot{x}[m\ddot{x} + \frac{dV}{dx}] = 0$$

□

For a general potential energy, using the conservation of energy law, we have:

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}(E - V(x))}$$

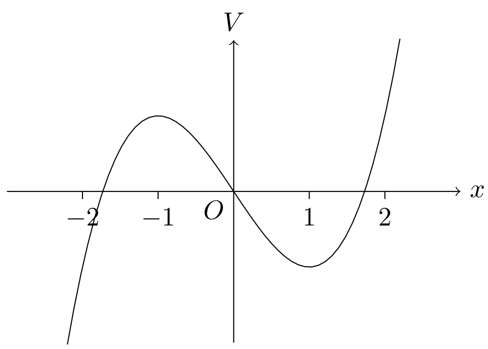
$$\Rightarrow t - t_0 = \pm\int_{x_0}^{x} \sqrt{\frac{2}{m}(E - V(x))}$$

To find $x(t)$ we need to do the integral and then solve for x (not usually possible by analytical methods).

3.2 Motion in a potential

The graph of the potential energy $V(x)$ gives us a qualitative understanding of the dynamics (does the particle escape?).

Example. $V(x) = m(x^3 - 3x)$



We release the particle from rest, then $E = V(x_0)$. At $x = \pm 1$, the particle stays there for all t. for $-1 < x_0 < 2$, we have a potential well (particle oscillates). If $x_0 < -1$, particle goes to $x = -\infty$. (It reaches this point in finite time) If $x_0 > 2$, the particle overshoots and continues to $x = -\infty$.

Now we have the special case $x_0 = 2$: particle goes to $x = -1$. how long does it take, and what happens next? We have:

$$KE = 2m - m(x^3 - 3x) = \frac{1}{2}m\dot{x}^2$$

$$\Rightarrow \frac{dt}{dx} = -\frac{1}{\sqrt{\frac{2}{m}(2m - m(x^3 - 3x))}}$$

Let $x = -1 + \epsilon(t)$.

$$\frac{2}{m}(E - V(x)) = 4 - 2(-1+\epsilon)^3 + 6(-1+\epsilon) = 6\epsilon^2 - 2\epsilon^3$$

$$t - t_0 = -\int_3^{\epsilon} \frac{d\epsilon}{\sqrt{6\epsilon^2 - 2\epsilon^3}}$$

Integrand $\propto \frac{1}{\epsilon}$ as $\epsilon \to 0$, so it takes infinite time to each $\epsilon = 0$.

3.3 Equilibrium Points

A particle placed at an *equilibrium point* x_0 will stay there for all time, and clearly the equilibrium points are the stationary points of potential energy (no force as $m\ddot{x} = -\frac{dV}{dx}$). Expanding V in a Taylor series and neglecting higher order terms, we have the equation of motion near the equilibrium points as :

$$m\ddot{x} \approx -V''(x_0)(x - x_0)$$

If $V''(x_0) > 0$ then V has a local minimum at x_0 and we have the potential of the harmonic oscillator (equilibrium point is stable). We have oscillations with

$$\omega = \sqrt{\frac{V''(x_0)}{m}}$$

If $V''(x_0) < 0$ then V has a local maximum at x_0 (unstable equilibrium), and we have solutions of exponential functions $x - x_0 \approx Ae^{\gamma t} + Be^{-\gamma t}$, $\gamma = \sqrt{\frac{-V''(x_0}{m}})$ for motion. For almost all initial conditions $A \neq 0$ and the particle will diverge from the equilibrium point.

3.4 Force and Potential Energy in three dimensions

Consider a particle of mass m moving in 3D. W have the equation of motion as:

$$m\ddot{\mathbf{r}} = \mathbf{F}$$

The kinetic energy is $T = \frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}m\dot{r}\cdot\dot{r}$, and it satisfies $\frac{dT}{dt} = m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}} = \mathbf{F}\cdot\dot{\mathbf{r}}$. So $\mathbf{F}\cdot\mathbf{v}$ is the *power*, the rate at which work is done on the particle by the force. The work done on the particle as it moves from $\mathbf{r}_1 = \mathbf{r}(t_1)$ to $\mathbf{r_2} = \mathbf{r}(t_2)$ along a trajectory C is the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = T(t_2) - T(t_1)$$

Suppose we have a force field $\mathbf{F}(\mathbf{r})$. A *conservative force* is one of the form $\mathbf{F} = -\nabla V$ where $V(\mathbf{r})$ is a *potential energy* functions. If $\mathbf{F}$ is conservative then the energy is conserved:

$$E = T + V = \frac{1}{2} m |v|^2 + V(\mathbf{r})$$

Proof.

$$\begin{aligned}\frac{dE}{dt} &= m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \frac{\partial V}{\partial x_i} \frac{dx_i}{dt} \\ &= (m\ddot{\mathbf{r}} + \nabla V) \cdot \dot{\mathbf{r}} = 0\end{aligned}$$

□

The work done is independent of the path:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C (\nabla V) \cdot d\mathbf{r} = V(\mathbf{r}_1) - V(\mathbf{r}_2)$$

3.5 Central forces

Definition. A special class of conservative forces has $V(r)$ with $r = |\mathbf{r}|$. Then

$$\mathbf{F} = -\frac{dV}{dr} \hat{\mathbf{r}}$$

where $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$ is a unit vector in the radial direction, pointing away from the origin. These forces are *central forces.*

Central forces has an additional conserved quantity called *angular momentum.*

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}} \qquad \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = m\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \mathbf{F} = \mathbf{0}$$

Definition. For a non-central force

$$\frac{d\mathbf{L}}{dt} = \mathbf{G}$$

where $\mathbf{G} = \mathbf{r} \times \mathbf{F}$ is the *torque.*

For a central force, only the angular momentum about center of force is conserved.

3.6 Gravity

Definition. Gravity is a conservative central force. If a particle of mass M is fixed at the origin then a second particle of mass m experiences a potential energy:

$$V(r) = -\frac{GMm}{r}$$

where G is the *gravitational constant.*Thus $\mathbf{F} = -\frac{GMm}{r^2} \hat{\mathbf{r}}$.

Definition. The *gravitational potential* is the gravitational potential energy per unit external mass (left). The *gravitational field* is the force per unit external mass (right).

$$\Phi_g(r) = -\frac{GM}{r}. \qquad \mathbf{g} = -\nabla\Phi_g = -\frac{GM}{r^2}\hat{\mathbf{r}}.$$

Proposition. The gravitational potential due t fixed masses M_i at points $\mathbf{r}_i$ is

$$\Phi_g(\mathbf{r}) = -\sum_i \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i|}.$$

Proposition. The external gravitational potential of a spherically symmetric object of mass M is equivalent to a point particle with the same mass at the center of object:

$$\Phi_g(r) = -\frac{GM}{r}.$$

Proof. cf. Vector Calculus □

Example. How fast do we need to jump to escape the gravitational pull of the Earth? If we jump upwards with speed v from the surface, then

$$E = T + V = \frac{1}{2}mv^2 - \frac{GMm}{R}.$$

Since after escape, we have $T \geq 0$ and $V = 0$, and energy is conserved, we must have $E \geq 0$ from the very beginning. ie.

$$v > v_{esc} = \sqrt{\frac{2GM}{R}}.$$

3.6.1 Inertial and Gravitational Mass

Mass appears in two equations:

$$m\ddot{\mathbf{r}} = \mathbf{F}$$
$$\mathbf{F} = -\frac{GM_g m_g}{r^2}\hat{\mathbf{r}}$$

Conceptually these are quite different, but experimentally, we have $m_i = m_g$ (to the accuracy of 10^{-12}) This is only explained by Einstein's theory of general relativity.

3.7 Electromagnetism

Definition. The EM force experienced by a particle with electric charge q is:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where $\mathbf{E}(r, t)$ is the electric field, $\mathbf{B}(r, t)$ is the magnetic field, and $\mathbf{v} = \dot{\mathbf{r}}$ is the velocity of the particle. This is the *Lorentz's force law.*

The electron has charge $q_e = -1.60 * 10^{-19}C$ and other particles' charges are integer multiples of this unit.

We only consider time-independent fields, so $\mathbf{E} = -\nabla\Phi_e$ where $\Phi_e(\mathbf{r})$ is the *electrostatic potential.* The electric force $q\mathbf{E}$ is then conservative.

Proposition. For time-independent $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, the energy $E = T + V = \frac{1}{2}m|\mathbf{v}|^2 + q\Phi_e$ is conserved and the magentic force does no work on the particle.

Proof.

$$\frac{dE}{dt} = m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} + q(\nabla\Phi_e) \cdot \dot{\mathbf{r}} = (m\ddot{\mathbf{r}} - q\mathbf{E}) \cdot \dot{\mathbf{r}} = (q\dot{\mathbf{r}} \times B) \cdot \dot{\mathbf{r}} = 0$$

□

3.7.1 Point charges

A particle of charge Q, fixed at the origin, produces an electrostatic potential:

$$\Phi_e = \frac{Q}{4\pi\epsilon_0 r}$$

and an electric field :

$$\mathbf{E} = -\nabla\Phi_e = \frac{Q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$$

Here $\epsilon_0 \approx 8.85 * 10^{-12} m^{-3} kg^{-1} s^2 C^2$ is the *electric constant.*

Definition. The resulting force,the *Coulomb force*, on a particle of charge q is:

$$\mathbf{F} = q\mathbf{E} = \frac{Qq}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$$

Electrostatic force is similar to gravity, but charges can be positive or negative so electrostatic force can be attractive or repulsive.

3.7.2 Motion in a uniform magnetic field

Set $\mathbf{E} = \mathbf{0}$ and choose axes such that $\mathbf{B} = (0, 0, B)$, a constant. According to the Lorentz force law $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, the components of the equation of motion are

$$m\ddot{x} = qB\dot{y}$$

$$m\ddot{y} = -qB\dot{x}$$

$$m\ddot{z} = 0$$

To solve this, we let $\zeta = x + iy$. Then $(1) + (2)i$ becomes $m\ddot{\zeta} = -iqB\dot{\zeta} \Rightarrow \zeta = \alpha e^{-i\omega t} + \beta$ where $\omega = \frac{qB}{m}$ is the *gyrofrequency* and α, β are complex constants.

Choose axes so at $t = 0$, $\mathbf{r} = \mathbf{0}$ and $\dot{\mathbf{r}} = (0, v, w)$. The solution is (helical motion):

$$x = R(1 - \cos\omega t, y = R\sin\omega t, z = wt$$

with $R = \frac{mv}{qB}$ the *gyro radius* or *Larmar radius.*

Vectorial treatment:

Let $\mathbf{B} = B\mathbf{n}$ with $|n| = 1$. Then $\ddot{\mathbf{r}} = \omega\dot{\mathbf{r}} \times \mathbf{n}$ with $\omega = \frac{qB}{m}$. Integrate once, we get $\dot{\mathbf{r}} = \omega\mathbf{r} \times \mathbf{n} + \mathbf{v}_0$ assuming $\mathbf{r}(0) = \mathbf{0}$ and $\dot{\mathbf{r}}(0) = \mathbf{v}_0$. Project this solution parallel and perpendicular to $\mathbf{B}$. First we get that $\dot{\mathbf{r}} \cdot \mathbf{n} = w$, which is $\mathbf{r} \cdot \mathbf{n} = wt$.Write $\mathbf{r} = (\mathbf{r} \cdot \mathbf{n})\mathbf{n} + \mathbf{r}_\perp$.

Then in the perpendicular component we have $\dot{\mathbf{r}_\perp} = \omega\mathbf{r}_\perp \times \mathbf{n} + \mathbf{n}_0 - (\mathbf{v}_0 \cdot \mathbf{n})\mathbf{n} \Rightarrow \ddot{\mathbf{r}_\perp} = -\omega^2\mathbf{r}_\perp + \omega\mathbf{v}_0 \times \mathbf{n}$ and solve using PI+CF.

3.8 Friction

Energy is conserved at an atomic level but does not appear to be conserved in many everyday processes. This is described by friction.

3.8.1 Dry Friction

When solid objects are in contact, a *normal reaction force* $\mathbf{N}$ ($\perp$ to the contact surface) prevents them from interpenetrating, while a frictional force $\mathbf{F}$ (tangential to the surface) resists relative tangential motion (sliding or slipping).

Definition. *Static friction* is the friction if no sliding occurs and has magnitude $|F| \leq \mu_s|\mathbf{N}|$, where μ_s is the coefficient of static friction. If sliding does occur, we have *Kinetic friction*, with magnitude $|\mathbf{F}| = \mu_k|\mathbf{N}|$ where μ_k is the coefficient of kinetic friction.

3.8.2 Fluid drag

Definition. When a solid object moves through a fluid it experiences a *drag force*. The two important regimes include *linear drag* ($\mathbf{F} = -k_1\mathbf{v}$, v is velocity of object relative to fluid, $k_1 = \sigma\pi\mu R > 0$ for a sphere of radius R) and *quadratic drag* ($\mathbf{F} = -k_1|\mathbf{v}|\mathbf{v}$ with other definitions same).

Example. Consider a projectile moving in a uniform gravitational field and experiencing a linear drag force.

We have the equation of motion for $m\frac{d\mathbf{v}}{dt} = m\mathbf{g} = k\mathbf{v}$. We solve this using an integrating factor and get $\mathbf{v} = \frac{m}{k}\mathbf{g} + \mathbf{c}e^{-\frac{kt}{m}}$. Then we substitute initial conditions ($\mathbf{v} = \mathbf{u}$ at $t = 0$) and integrate again to get:

$$\mathbf{x} = \frac{m}{k}\mathbf{g}t - \frac{m}{k}(\mathbf{u} - \frac{m}{k}\mathbf{g})e^{-\frac{kt}{m}} + \mathbf{d}$$

Then we substitute further initial conditions of $x = 0$ and $t = 0$ to get:

$$\mathbf{x} = \frac{m}{k}\mathbf{g}t + \frac{m}{k}(\mathbf{u} - \frac{m}{k}(1 - \mathbf{g})e^{-\frac{kt}{m}})$$

In components:

$$x = \frac{mu}{k}\cos\theta(1 - e^{-\frac{kt}{m}}), \text{ and } z = -\frac{mgt}{k} + \frac{m}{k}(u\sin\theta + \frac{mg}{k})(1 - e^{-\frac{kt}{m}})$$

3.8.3 Effect of damping on small oscillations

Friction or drag causes oscillations about a potential minimum to be damped out. Energy is continually lost until the system comes to rest at the stable equilibrium. For details, refer to Differential Equations.

4 Orbits

We will study the motion in 3D of a particle in a central force,

$$m\ddot{\mathbf{r}} = -\nabla V(r).$$

The angular momentum $\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}$ is a constant vector, as previously shown. Furthermore $\mathbf{L} \cdot \mathbf{r} = 0$. Therefore, the motion takes place in a plane passing through the origin, and perpendicular to $\mathbf{L}$.

4.1 Polar coordinates in the plane

We choose our axes such that the orbital plane is $z = 0$. We introduce polar coordinates (r, θ):

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Definition. We define unit vectors in direction of increasing r and increasing θ:

$$\hat{\mathbf{r}} = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}, \quad \hat{\boldsymbol{\theta}} = \begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}.$$

They form an orthonormal basis at each point, but depends on position:

Proposition.

$$\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}\theta} = \begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix} = \hat{\boldsymbol{\theta}} \qquad \frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}\theta} = \begin{pmatrix}-\cos\theta \\ -\sin\theta\end{pmatrix} = -\hat{\mathbf{r}}.$$

Proposition. For each particle with trajectory $\mathbf{r}(t)$, the polar coordinates r and θ depend on t, and so do the polar unit vectors. By the chain rule,

$$\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} = \dot{\theta}\hat{\boldsymbol{\theta}}, \quad \frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t} = -\dot{\theta}\hat{\mathbf{r}}.$$

In terms of the polar unit vectors,

$$\mathbf{r} = r\hat{\mathbf{r}}.$$

The velocity is

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}.$$

The acceleration is

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.\end{aligned}$$

Definition. $\dot{r}$ is the *radial velocity*, and $\dot{\theta}$ is the *angular velocity*.

4.2 Motion in a central force field

Since $V = V(r)$, $\mathbf{F} = -\nabla V = \frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}}$.

So Newton's 2nd law in polar coordinates is

$$m(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}}.$$

The θ component of this equation is

$$m(r\ddot{\theta} - 2\dot{r}\dot{\theta}) = 0.$$

So

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = 0.$$

Write $L = mr^2\dot{\theta}$. This is saying conserved angular momentum $\mathbf{L}$:

$$\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}} = mr\hat{\mathbf{r}} \times (\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) = mr^2\dot{\theta}\ \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = mr^2\dot{\theta}\ \hat{\mathbf{z}}.$$

So are above equation states that L is constant, which is consistent with the conservation of angular momentum.

Definition. $h = \frac{L}{m} = r^2\dot{\theta}$, *angular momentum per unit mass* is constant.

The radial component of the equation of motion is:

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{dV}{dr}$$

Eliminate $\dot{\theta}$ using $r^2\dot{\theta} = h$:

$$m\ddot{r} = -\frac{dV}{dr} + \frac{mh^2}{r^3} = -\frac{dV_{eff}}{dr}$$

where $V_{eff}(r) = V(r) + \frac{mh^2}{2r^2}$ is the effective potential energy.

Now we have reduced the problem to 1D motion.

The energy of the particle is:

$$E = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + V(r) = \frac{1}{2}m(\dot{r}^2 + r^2(\dot{\theta})^2) + V(r) = \frac{1}{2}m\dot{r}^2 + V_{eff}(r)$$

- If $E = E_{min}$ then r remains at r_* and it does uniform motion in a circle.
- If $E_{min} < E < 0$ then r oscillates and $\dot{r} = \frac{h}{r^2}$ does also. This is a non-circular, bound orbit.

  Definition. The points of minimum and maximum r in such an orbit are called the *periapsis* and *apoapsis*, collectively known as *apsides.*. For Earth/non-Earth's orbit around the sun, periapsis and apoapsis are called *perigee/perihelion* and *aphelion/apogee.*

- If $E \geq 0$, then r comes in from ∞, reaches a minimum and returns to ∞. This is an unbound orbit.

4.3 Stability of circular Orbits

Now consider a general potential energy $V(r)$. Do circular orbits exist, and are they stable? For a circular orbit, $r = r_* = k$ for some value of $h \neq 0$. Since $\ddot{r} = 0$, we require $V'_{eff}(r_*) = 0$

The orbit is stable if r_* is a minimum of V_{eff}:

$$V''_{eff}(r_*) > 0$$

In terms of $V(r)$, we require $V'(r_*) = \frac{mh^2}{r_*^3}$ for a circular orbit and it is stable if:

$$V''(r_*) + \frac{3mh^2}{r_*^4} = V''(r_*) + \frac{3}{r_*}V'(r_*) > 0 \text{ or } F'(r_*) + \frac{3}{r}F(r_*) < 0$$

In terms of the radial force $F(r) = -V'(r)$.

4.4 Equation for the shape of the orbit

$$t = \pm\sqrt{\frac{m}{2}} \int \frac{dr}{\sqrt{E - V_{eff}(r)}}$$

but this is not usually practical. It is much easier to find the shape $r(\theta)$ of the orbit by introducing the new variable:

$$u = \frac{1}{r}$$

Then we have:

$$\ddot{r} = -h^2 u^2 \frac{d^2 u}{d\theta^2}$$

The radial equation of motion becomes:

$$-mh^2 u^2 (\frac{d^2 u}{d\theta^2} + u) = F(\frac{1}{u})$$

This is known as Binet's Equation.

Given $F(r)$, we aim to sole this second-order ODE for $u(\theta)$. If needed, we can then work out the time-dependence via:

$$\dot{\theta} = hu^2$$

4.5 The Kepler problem

4.5.1 Shapes of orbits

For a planet orbiting the sun,

$$V(r) = \frac{mk}{r}, \quad F(r) = -\frac{mk}{r^2}$$

with $k = GM$. (For opposite charges' attraction, we have the same equation with $k = -\frac{Qq}{4\pi\epsilon_0 m}$.)

Binet's equation then becomes linear, and we write the general solution as

$$u = \frac{k}{h^2} + A\cos(\theta - \theta_0),$$

where $A \geq 0$ and θ_0 are arbitrary constants.

If $A = 0$, then u is constant, and the orbit is circular. Otherwise, we redefine polar coordiantes so that u reaches a maximum at $\theta = 0$ (periapsis). Then:

Proposition. The orbit of a planet around the sun is given by

$$r = \frac{\ell}{1 + e\cos\theta}, \tag{$*$}$$

with $\ell = h^2/k$ and $e = Ah^2/k$. This is a conic (in polars), with the sun, a focus, at the origin.

Definition (Eccentricity)**.** The dimensionless parameter $e \geq 0$ in the equation of orbit is the *eccentricity* and determines the shape of the orbit.

($*$) in Cartesian coordinates with $x = r\cos\theta$ and $y = r\sin\theta$ is:

$$(1 - e^2)x^2 + 2e\ell x + y^2 = \ell^2. \tag{$\dagger$}$$

There are three different possibilities:

- Ellipse: ($0 \le e < 1$). r is bounded by

$$\frac{\ell}{1+e} \le r \le \frac{\ell}{1-e}.$$

  (†) after reordering is an ellipse with center $(-ea, 0)$:

$$\frac{(x+ea)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

  where $a = \dfrac{\ell}{1-e^2}$ and $b = \dfrac{\ell}{\sqrt{1-e^2}} \le a$.

  a and b are the semi-major and semi-minor axis. ℓ is the *semi-latus rectum.* One focus is at the origin. If $e = 0$, then $a = b = \ell$ and the ellipse is a circle.

- Hyperbola: ($e > 1$). For $e > 1$, $r \to \infty$ as $\theta \to \pm\alpha$, where $\alpha = \cos^{-1}(1/e) \in (\pi/2, \pi)$. Then (†) rearranges to the equation of a hyperbola centered on $(ea, 0)$,

$$\frac{(x-ea)^2}{a^2} - \frac{y^2}{b^2} = 1,$$

  with $a = \dfrac{\ell}{e^2-1}$, $b = \dfrac{\ell}{\sqrt{e^2-1}}$.

  This corresponds to an unbound orbit that is deflected by an attractive force.

  b is both the semi-minor axis and the *impact parameter.* It is the distance by which the planet would miss the object if there were no attractive force.

  The asymptote is $y = \frac{b}{a}(x - ea)$, or $x\sqrt{e^2-1} - y = eb$.

- Parabola: ($e = 1$). We see that $r \to \infty$ as $\theta \to \pm\pi$. (†) becomes the equation of a parabola, $y^2 = \ell(\ell - 2x)$. The trajectory is similar to that of a hyperbola.

4.5.2 Energy and eccentricity

We can figure out which path a planet follows by considering its energy.

$$E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}) - \frac{mk}{r} = \frac{1}{2}mh^2\left(\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) + u^2\right) - mku$$

Substitute $u = \dfrac{1}{\ell}(1 + e\cos\theta)$ and $\ell = \dfrac{h^2}{k}$, it simplifies to (does not depend on θ)

$$E = \frac{mk}{2\ell}(e^2 - 1),$$

So bounded orbits have $e < 1$, and thus $E < 0$. Unbounded orbits have $e > 1$ and $E > 0$. A parabolic orbit has $e = 1$, $E = 0$, and is "marginally bound".

The condition $E > 0$ is equivalent to $|\dot{\mathbf{r}}| > \sqrt{\frac{2GM}{r}} = v_{\text{esc}}$, which means you have enough kinetic energy to escape orbit.

4.5.3 Kepler's laws of planetary motion

Law (First law)**.** The orbit of each planet is an ellipse with the Sun at one focus.

Law (Second law)**.** The line between the planet and the sun sweeps out equal areas in equal times.

Law (Third law)**.** The square of the orbital period is proportional to the cube of the semi-major axis, or

$$P^2 \propto a^3.$$

- Law 1 follows from Newtonian dynamics (as shown earlier).
- Law 2 is true for any central force, as it is the conservation of angular momentum: The area swept out by moving $\mathrm{d}\theta$ is $\mathrm{d}A = \frac{1}{2}r^2 \,\mathrm{d}\theta$ (area of sector of circle). So
$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \frac{h}{2} = \text{const.}$$
- Law 3 follows from this: the total area of the ellipse is $A = \pi ab = \frac{h}{2}P$ (by the second law). But $b^2 = a^2(1-e^2)$ and $h^2 = k\ell = ka(1-e^2)$. So
$$P^2 = \frac{(2\pi)^2 a^4(1-e^2)}{ka(1-e^2)} = \frac{(2\pi)^2 a^3}{k}.$$

4.6 Rutherford scattering

Consider motion in a *repulsive* inverse-square force,

$$V(r) = +\frac{mk}{r}, \quad F(r) = +\frac{mk}{r^2}.$$

The solution is now

$$u = -\frac{k}{h^2} + A\cos(\theta - \theta_0).$$

We can take $A \geq 0, \theta_0 = 0$ wlog. Then

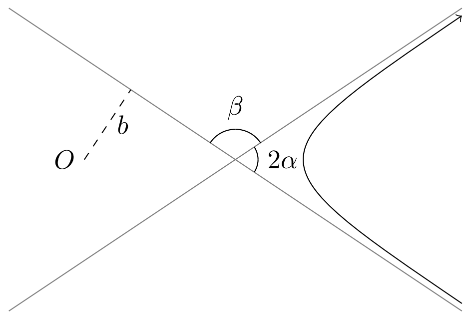
$$r = \frac{\ell}{e\cos\theta - 1}, (\ell = \frac{h^2}{k}, \quad e = \frac{Ah^2}{k})$$

r and ℓ are positive, so $e \geq 1$. Then $r \to \infty$ as $\theta \to \pm\alpha$, where $\alpha = \cos^{-1}(1/e)$.

The orbit is a hyperbola. Again,

$$\frac{(x-ea)^2}{a^2} - \frac{y^2}{b^2} = 1, (a = \frac{\ell}{e^2-1}, b = \frac{\ell}{\sqrt{e^2-1}})$$

Far from origin, the orbit tends to uniform linear motion with speed v. Then angular momentum per unit mass is $h = bv$ (velocity $\times$ perpendicular distance to O.



How does the scattering angle $\beta = \pi - 2\alpha$ depend on the impact parameter b and the incident speed v? From above,

$$\frac{1}{e} = \cos\alpha = \cos\left(\frac{\pi}{2} - \frac{\beta}{2}\right) = \sin\left(\frac{\beta}{2}\right),$$

So

$$b = \frac{\ell}{\sqrt{e^2 - 1}} = \frac{(bv)^2}{k}\tan\frac{\beta}{2} \Rightarrow \beta = 2\tan^{-1}\left(\frac{k}{bv^2}\right).$$

So scattering angles can be obtained for small impact parameters, $b \ll k/v^2$.

5 Rotating frames

In a rotating frame (non-inertial), Newton's laws need to be changed. Let S be an inertial frame, and let S' be a non-inertial frame, rotating about the z axis with angular velocity $\omega = \dot{\theta}$ with respect to S.

Call the basis vectors $\mathbf{e}_i = \{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ and $\mathbf{e}'_i = \{\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'\}$.

Consider a particle at rest in $\mathbf{S}'$. From the perspective of S, its velocity is

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_S = \boldsymbol{\omega} \times \mathbf{r},$$

where $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ is the *angular velocity vector* (aligned with the rotation axis). This formula also applies to the basis vectors of S'.

$$\left(\frac{\mathrm{d}\mathbf{e}'_i}{\mathrm{d}t}\right)_S = \boldsymbol{\omega} \times \mathbf{e}'_i.$$

A general time-dependent vector $\mathbf{a}$ can be written as

$$\mathbf{a} = \sum a'_i(t)\mathbf{e}'_i.$$

From the perspective of S', with $\mathbf{e}'_i$ constant and its time derivative is

$$\left(\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}\right)_{S'} = \sum \frac{\mathrm{d}a'_i}{\mathrm{d}t}\mathbf{e}'_i.$$

In S, $\mathbf{e}'_i$ is not constant, so its time derivative is (this is true for any vector $\mathbf{a}$):

$$\left(\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}\right)_S = \sum \frac{\mathrm{d}a_i}{\mathrm{d}t}\mathbf{e}'_i + \sum a'_i\boldsymbol{\omega} \times \mathbf{e}'_i = \left(\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{a}.$$

So the difference in velocity measured in the two frames is the relative velocity of the frames, which depends on position.

Applied a second time, and allowing for $\boldsymbol{\omega}$ to depend on time, it gives

$$\begin{aligned}\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_S &= \left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times\right)\left(\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{r}\right). \\ &= \left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\end{aligned}$$

Since S is inertial, Newton's Second Law is

$$m\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_S = \mathbf{F}.$$

Therefore, Newton's second law is transformed to:

$$m(\frac{d^2\mathbf{r}}{dt^2})_{S'} = \mathbf{F} - 2m\boldsymbol{\omega} \times (\frac{d\mathbf{r}}{dt})_{S'} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

The additional terms on the RHS are called *fictitious forces*. They are:

Definition. The *Coriolis force*, $-2m\boldsymbol{\omega} \times (\frac{d\mathbf{r}}{dt})_{s'}$, the *Euler force*, -$m\dot{\boldsymbol{\omega}} \times \mathbf{r}$, the *Centrifugal force* $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. Usually we consider $\boldsymbol{\omega}$ as constant so we ignore the Euler force.

A frame fixed to the Earth rotates with angular velocity:

$$|\boldsymbol{\omega}| = \frac{2\pi}{1 \text{ day}} \approx 7.3 * 10^{-5} s^{-1}$$

5.0.1 The centrifugal force

Let $\boldsymbol{\omega} = \omega\hat{\boldsymbol{\omega}}$ where $|\hat{\boldsymbol{\omega}}| = 1$. Then:

$$-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = m\omega^2\mathbf{r}_\perp$$

where $\mathbf{r}_\perp$ is the projection of $\mathbf{r}$ onto the plane perpendicular to $\boldsymbol{\omega}$.

Note that $\mathbf{r}_\perp \cdot \mathbf{r}_\perp = \mathbf{r} \cdot \mathbf{r} - (\mathbf{r} \cdot \hat{\boldsymbol{\omega}})^2$

$$\nabla(|\mathbf{r}_\perp|^2) = 2\mathbf{r} - 2\mathbf{r} \cdot \hat{\boldsymbol{\omega}}\hat{\boldsymbol{\omega}} = 2\mathbf{r}_\perp$$

So the centrifugal force is a potential force. Then we have (from above):

$$-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -\nabla(-\frac{1}{2}m\omega^2|\mathbf{r}_\perp|^2)$$

On a rotating planet, the gravitational and centrifugal forces per unit mass combine to make the effective gravity.

$$\mathbf{g}_{\text{eff}} = \mathbf{g} + \omega^2\mathbf{r}_\perp.$$

Note. $\mathbf{g}_{\text{eff}}$ does not point to the ground!

5.0.2 The Coriolis Force

This depends on the velocity in the rotating frame, $\mathbf{v} = (\frac{d\mathbf{r}}{dt})_{s'}$

Like the lorentz force in a magnetic field, it is always perpendicular to $\mathbf{v}$ so it does no work on the particle. Consider motion parallel to the Earth's surface.

The vertical $(\mathbf{z})$ component of $\boldsymbol{\omega}$, $\omega \sin\lambda\hat{\mathbf{z}}$, combined with the horizontal velocity $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$ generates horizontal Coriolis force

$$-2m\omega \sin\lambda\hat{\mathbf{z}} \times \mathbf{v} = 2m\omega \sin\lambda(v_y\hat{\mathbf{x}} - v_x\hat{\mathbf{y}})$$

in the Northern Hemisphere. This causes a deflection towards the right. in the Southern Hemisphere the deflection is to the left. The effect vanishes at the equator.

The Coriolis force is most important for large-scale terrestrial motions (atmosphere and oceanic flows) with time-scales comparable to a day. (such as cyclones)

Suppose a ball is dropped from a tower at height h at the equator. where does it land? In the rotating frame,

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} - \boldsymbol{\omega} \times (\omega \times \mathbf{r})$$

We work to first order in ω. Then:

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} - O(\omega^2)$$

Integrate:

$$\dot{\mathbf{r}} = \mathbf{g}t - 2\boldsymbol{\omega} \times \mathbf{r} - \mathbf{r_0} - O(\omega^2)$$

Substitute, Integrate, and use $\mathbf{g} = (0, 0, -g)$, $\boldsymbol{\omega} = (0, \omega, 0)$ and $\mathbf{r}_0 = (0, 0, R + h)$:

$$\mathbf{r} = \mathbf{r}_0 + \frac{1}{2}\mathbf{g}t^2 - \frac{1}{3}\omega \times \mathbf{g}t^3 + O(\omega^2) \Rightarrow \mathbf{r} = (\frac{1}{3}\omega g t^3, 0, R + h - \frac{1}{2}gt^2)$$

When we omit the centrifugal force. So the particle hits the ground at $t = \sqrt{\frac{2h}{g}}$ and its eastward displacement is $\frac{1}{3}\omega g(\frac{2h}{g})^{\frac{3}{3}}$.

This can be understood in terms of angular momentum conservation in the non-rotating frame. The eastward velocity increases as the distance from the centre of the Earth decreases.

5.1 Foucault's Pendulum

Consider a pendulum that is free to swing in any plane. At the North Pole, it will swing in a plane that is fixed in an inertial frame, while the Earth rotates beneath it. From the perspective of the rotating frame, the plane of the pendulum rotates backwards due to the Coriolis force. At latitude λ the plane rotates with period $\frac{1}{\sin\lambda}$ day.

6 System of particles

Consider a system of N interacting particles. Particle i has mass m_i, position $\mathbf{r}_i$, and momentum $f_i = m_i\dot{\mathbf{r}}_i$. Newton's second law for particle i is:

$$m_i\ddot{\mathbf{r}} = \dot{\mathbf{p}}_i = \mathbf{F}_i \qquad (\mathbf{F}_i = \mathbf{F}_i^{ext} + \sum_{j=1}^{N} \mathbf{F}_{ij})$$

$\mathbf{F}_i^{ext}$ is the external force on particle i from particles outside the system. $\mathbf{F}_{ij}$ is the force on particle i due to particle j. Now we can state Newton's third law:

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}$$

6.1 Motion of the centre of mass

Definition. The total *mass* of the system is $M = \sum_{i=1}^{N} m_i$ The *centre of mass*, or mass-weighted average position, is located at:

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{N} m_i\mathbf{r}_i$$

The total linear momentum is:

$$\mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i = M\dot{\mathbf{R}}$$

equivalent to that of a single particle of mass M at the centre of mass. Then

$$M\ddot{\mathbf{R}} = \dot{\mathbf{P}} = \sum_{i=1}^{N} \dot{\mathbf{p}}_i = \sum_{i=1}^{N} \mathbf{F}_i^{ext} + \sum_{i=1}^{N}\sum_{j=1}^{N} \mathbf{F}_{ij} = \mathbf{F} + \frac{1}{2}\sum_i\sum_j (\mathbf{F}_{ij} + \mathbf{F}_j i) = \mathbf{F}$$

where $\mathbf{F} = \sum_i \mathbf{F}_i^{ext}$ is the total external force. The internal forces cancel, by Newton's Third Law. So the centre of mass moves as a particle of mass M, subject to a force $\mathbf{F}$. This is why newton's laws apply to macroscopic objects as well as individual particles.

6.1.1 Conservation of momentum

If $\mathbf{F} = \mathbf{0}$ then $\dot{\mathbf{P}} = \mathbf{0}$: the total momentum is conserved. Then there is an inertial "centre of mass frame" in which $\mathbf{R} = \mathbf{0}$ for all t.

6.1.2 Conservation of Angular momentum

Definition. 2 The total angular momentum of the system about the origin is:

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i$$

So:

$$\dot{\mathbf{L}} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i \mathbf{G}_i^{ext} + \frac{1}{2}\sum_i\sum_j (\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji}) = \mathbf{G} + \frac{1}{2}\sum_i\sum_j (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}$$

where $\mathbf{G} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext}$ is the total external torque about the origin. Provided that the strong version of Newton's third law holds:

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji} \text{ and is parallel to } (\mathbf{r}_i - \mathbf{r}_j)$$

If there is no external torque, the angular momentum is conserved (true for gravitational and electrostatic forces).

6.2 Motion relative to the centre of mass

Let $\mathbf{r}_i = \mathbf{R} + \mathbf{r}_i^c$ where $\mathbf{r}_i^c$ is the position of particle i relative to the centre of mass. Then:

$$\sum_i m_i \mathbf{r}_i^c = \sum_i m_i \mathbf{r}_i - \sum_i m_i \mathbf{R} = M\mathbf{R} - M\mathbf{R} = 0 \Rightarrow \sum_i m_i \dot{\mathbf{r}_i^c} = 0$$

The total linear momentum, angular momentum and kinetic energy are:

$$\mathbf{P} = \sum_i m_i (\dot{\mathbf{R}} + \dot{\mathbf{r}}_i^c = M\dot{\mathbf{R}}$$

$$\mathbf{L} = \sum_i m_i (\mathbf{R} + \mathbf{r}_i^c) \times (\dot{\mathbf{R}} + \dot{\mathbf{r}}_i^c)$$

$$= \sum_i m_i \mathbf{R} \times \dot{\mathbf{R}} + \mathbf{R} \times \sum_i m_i \dot{\mathbf{r}_i}^c + \sum_i m_i \mathbf{r}_i^c \times \dot{\mathbf{R}} + \sum_i m_i \mathbf{r}_i^c \times \dot{\mathbf{r}}_i^c$$

$$= M\mathbf{R} \times \dot{\mathbf{R}} + \sum_i m_i \mathbf{r}_i^c \times \dot{\mathbf{r}}_i^c$$

$$T = \frac{1}{2}\sum_i m_i |\dot{\mathbf{r}}_i|^2 = \frac{1}{2}\sum_i m_i (\dot{\mathbf{R}} + \dot{\mathbf{r}_i}^c) \cdot (\dot{\mathbf{R}} + \dot{\mathbf{r}}_i^c) = \frac{1}{2}M|\dot{\mathbf{R}}|^2 + \frac{1}{2}\sum_i m_i |\dot{\mathbf{r}}_i^c|^2$$

If the forces are conservative in the sense that:

$$\mathbf{F}_i^{ext} = -\nabla_i V_i(\mathbf{r}_i) \quad \& \quad \mathbf{F}_{ij} = -\nabla_i V_{ij}(\mathbf{r}_i - \mathbf{r}_j)$$

where ∇_i is the gradient with respect to $\mathbf{r}_i$, then energy is conserved in the form:

$$E = T + \sum_i V_i(\mathbf{r}_i) + \frac{1}{2}\sum_i \sum_j V_{ij}(\mathbf{r}_i - \mathbf{r}_j)$$

6.3 The two-body problem

Consider two particles with no external forces. The centre of mass is at:

$$\mathbf{R} = \frac{1}{M}(m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2)$$

where $M = m_1 + m_2$. Define the separation vector (relative position vector):

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

Since $\mathbf{F} = 0$, $\ddot{\mathbf{R}} = 0$: the centre of mass moves uniformly. Meanwhile, $\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = (\frac{1}{m_1} + \frac{1}{m_2})\mathbf{F}_{12}$ Thus $\mu\ddot{\mathbf{r}} = \mathbf{F}_{12}(\mathbf{r})$. where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass.

Example. With gravity, we have:

$$\mu\ddot{\mathbf{r}} = -\frac{Gm_1 m_2 \hat{\mathbf{r}}}{|r|^2} \quad \& \quad \ddot{\mathbf{r}} = -\frac{G(m_1 + m_2)\hat{\mathbf{r}}}{|r|^2}$$

The orbital period depends on the total mass of the system, however for the solar system it is basically the mass of the sun. It can be shown that:

$$\mathbf{L} = M\mathbf{R} \times \dot{\mathbf{R}} + \mu\mathbf{r} \times \dot{\mathbf{r}} \quad \& \quad T = \frac{1}{2}M|\dot{\mathbf{r}}|^2 + \frac{1}{2}\mu|\ddot{\mathbf{r}}|^2$$

6.4 Variable-mass problems

Rockets, fireworks, and others have decreasing mass with respect to time.

Consider a rocket moving in one dimension with mass $m(t)$ and velocity $v(t)$. The rocket burnes fuel and ejecting the exhaust at velocity $-u$ relative to the rocket. Its initial momentum is $m(t)v(t)$, final momentum of rocket is $m(t+\delta t)v(t+\delta t)$, while the momentum of the exhaust is $[m(t) - m(t+\delta t][v(t) - u + O(\delta t)]$. So The change in total momentum of the system from t to $t + \delta t$ is:

$$\delta p = m(t+\delta t)v(t+\delta t) + [m(t) - m(t+\delta t][v(t) - u + O(\delta t)] - m(t)v(t)$$

$$= (m\dot{v} + \dot{m}u)\delta t + O(\delta t^2)$$

Newton's second law gives:

$$\frac{dp}{dt} = F$$

So, we have:

$$\frac{\delta p}{\delta t} = m\dot{v} + \dot{m}v + \frac{O(\delta t^2)}{\delta t} \Rightarrow m\frac{dv}{dt} + u\frac{dm}{dt} = F$$

7 Rigid bodies

Definition. A *rigid body* is an extended object, consisting of N particles that are constrained such that the distance $|\mathbf{r}_i - \mathbf{r}_j|$ between any two particles is fixed. The possible motions of a rigid body are the continuous isometries of Euclidean space, *translations* and *rotations*.

7.1 Angular velocity

Consider a single particle moving in a circle of radius s about the z axis. Its position and velocity vectors are

$$\mathbf{r} = (s\cos\theta, s\sin\theta, z)$$
$$\dot{\mathbf{r}} = (-s\dot{\theta}\sin\theta, s\dot{\theta}\cos\theta, 0).$$

We can write

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}, \qquad (\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{z}} \text{ is the angular velocity vector})$$

In general,

$$\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{n}} = \omega\hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is a unit vector parallel to the rotation axis.

The kinetic energy of this particle is

$$T = \frac{1}{2}m|\dot{\mathbf{r}}|^2 = \frac{1}{2}ms^2\dot{\theta}^2 = \frac{1}{2}I\omega^2$$

where $I = ms^2$ is the *moment of inertia* and s the distance from the particle to the axis of rotation.

7.2 Moment of inertia

Consider a rigid body in which all N particles rotate about the same axis with the same angular velocity:

$$\dot{\mathbf{r}}_i = \boldsymbol{\omega} \times \mathbf{r}_i.$$

This ensures that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{r}_i - \mathbf{r}_j|^2 = 2(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j)\cdot(\mathbf{r}_i - \mathbf{r}_j) = 2\big(\boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{r}_j)\big)\cdot(\mathbf{r}_i - \mathbf{r}_j) = 0,$$

The rotational kinetic energy is

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i|\dot{\mathbf{r}}_i|^2 = \frac{1}{2}I\omega^2,$$

where

Definition. The *moment of inertia* of a rigid body about the rotation axis $\hat{\mathbf{n}}$ (left) and the *angular momentum* (right) is

$$I = \sum_{i=1}^{N} m_i s_i^2 = \sum_{i=1}^{N} m_i |\hat{\mathbf{n}} \times \mathbf{r}_i|^2. \quad \& \quad \mathbf{L} = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i = \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i).$$

The component of $\mathbf{L}$ in the direction of $\hat{\mathbf{n}}$ is

$$\mathbf{L} \cdot \hat{\mathbf{n}} = \omega \sum_i m_i \hat{\mathbf{n}} \cdot (\mathbf{r}_i \times (\hat{\mathbf{n}} \times \mathbf{r}_i)) = \omega \sum_i m (\hat{\mathbf{n}} \times \mathbf{r}_i) \cdot (\hat{\mathbf{n}} \times \mathbf{r}_i) = I\omega$$

However, $\mathbf{L}$ may not be parallel to to $\boldsymbol{\omega}$ in general. Using vector identities, we have

$$\mathbf{L} = \sum_i m_i \big((\mathbf{r}_i \cdot \mathbf{r}_i) \boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega}) \mathbf{r}_i \big)$$

Note that this is a linear function of $\boldsymbol{\omega}$. So we can write

$$\mathbf{L} = I\boldsymbol{\omega}$$

where I is the *inertia tensor.* (cf Vector Calculus)

If the body rotates about a *principal axis*, ie. one of the three orthogonal eigenvectors of I, then $\mathbf{L}$ is parallel to $\boldsymbol{\omega}$.

The relations $T = \frac{1}{2} I\omega^2$ and $L = I\omega$ for angular motion are analogous to the relations $T = \frac{1}{2}mv^2$ and $p = mv$ for linear motion.

7.3 Calculating the moment of inertia

For a solid body, we usually want to think of it as a continuous substance with a mass density, instead of individual point particles. So we replace the sum of particles by a volume integral weighted by the mass density $\rho(\mathbf{r})$.

Definition. The *mass* (left), *center of mass* (middle), *moment of inertia* (right) are:

$$M = \int \rho \, \mathrm{d}V. \quad \& \quad \mathbf{R} = \frac{1}{M} \int \rho \mathbf{r} \, \mathrm{d}V \quad \& \quad I = \int \rho s^2 \, \mathrm{d}V = \int \rho |\hat{\mathbf{n}} \times \mathbf{r}|^2 \, \mathrm{d}V.$$

In theory, we can study inhomogeneous bodies with varying ρ, but usually we consider homogeneous ones with constant ρ throughout.

Example (Thin circular ring)**.** Suppose the ring has mass M and radius a, and a rotation axis through the center, perpendicular to the plane of the ring. Then the moment of inertia is

$$I = Ma^2.$$

Example (Thin rod)**.** Suppose a rod has mass M and length ℓ. It rotates through one end, perpendicular to the rod. The mass per unit length is M/ℓ. So the moment of inertia is

$$I = \int_0^\ell \frac{M}{\ell} x^2 \, \mathrm{d}x = \frac{1}{3} M\ell^2.$$

Theorem (Perpendicular axis theorem)**.** For a two-dimensional object (a lamina), and three perpendicular axes x, y, z through the same spot, with z normal to the plane,

$$I_z = I_x + I_y,$$

where I_z is the moment of inertia about the z axis.

Note. this does NOT apply to 3D objects! For example, in a sphere, $I_x = I_y = I_z$.

Proof. Let ρ be the mass per unit volume. Then

$$I_x = \int \rho y^2 \;\mathrm{d}A, I_y = \int \rho x^2 \;\mathrm{d}A \Rightarrow I_z = \int \rho(x^2 + y^2) \;\mathrm{d}A = I_x + I_y.$$

□

Example. For a disc, $I_x = I_y$ by symmetry. So $I_z = 2I_x$.

Theorem (Parallel axis theorem)**.** If a rigid body of mass M has moment of inertia I^C about an axis passing through the center of mass, then its moment of inertia about a parallel axis a distance d away is

$$I = I^C + Md^2.$$

Proof. With a convenient choice of Cartesian coordinates such that the center of mass is at the origin and the two rotation axes are $x = y = 0$ and $x = d, y = 0$,

$$I^C = \int \rho(x^2 + y^2) \;\mathrm{d}V, \quad \& \quad \int \rho\mathbf{r} \;\mathrm{d}V = \mathbf{0}.$$

So

$$I = \int \rho((x-d)^2 + y^2) \;\mathrm{d}V = \int \rho(x^2+y^2) \;\mathrm{d}V - 2d\int \rho x \;\mathrm{d}V + \int d^2\rho \;\mathrm{d}V = I^c + Md^2.$$

□

Example. Take a disc of mass M and radius a, and rotation axis through a point on the circumference, perpendicular to the plane of the disc. Then

$$I = I^c + Ma^2 = \frac{1}{2}Ma^2 + Ma^2 = \frac{3}{2}Ma^2.$$

7.4 Motion of a rigid body

The general motion of a rigid body can be described as a translation of its centre of mass, following a trajectory $\mathbf{R}(t)$, together with a rotation about an axis through the center of mass. We write

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_i^c. \quad \& \quad \dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{\mathbf{r}}_i^c.$$

Using this, we can break down the velocity and kinetic energy into translational and rotational parts. If the body rotates with angular velocity $\boldsymbol{\omega}$ about the center of mass, then

$$\dot{\mathbf{r}}_i^c = \boldsymbol{\omega} \times \mathbf{r}_i^c.$$

Since $\mathbf{r}_i^c = \mathbf{r}_i - \mathbf{R}$, we have

$$\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \boldsymbol{\omega} \times \mathbf{r}_i^c = \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{R}).$$

On the other hand, the kinetic energy (calculated in previous lectures) is

$$T = \frac{1}{2}M|\dot{\mathbf{R}}|^2 + \frac{1}{2}\sum_i m_i|\dot{\mathbf{r}}_i^c|^2 = \underbrace{\frac{1}{2}M|\dot{\mathbf{R}}|^2}_{\text{translational KE}} + \underbrace{\frac{1}{2}I^c\omega^2}_{\text{rotational}KE}.$$

Sometimes we do not want to use the center of mass as the center. For example, if an item is held at the edge and spun around, we'd like to study the motion about the point at which the item is held, and not the center of mass.

So consider any point Q, with position vector $\mathbf{Q}(t)$ that is not the center of mass but moves with the rigid body, ie.

$$\dot{\mathbf{Q}} = \dot{\mathbf{R}} + \boldsymbol{\omega} \times (\mathbf{Q} - \mathbf{R}).$$

Usually this is a point inside the object itself, but we do not assume that in our calculation.Then we can write

$$\begin{aligned}\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{R}) &= \dot{\mathbf{Q}} - \boldsymbol{\omega} \times (\mathbf{Q} - \mathbf{R}) + \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{R}) \\ &= \dot{\mathbf{Q}} + \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{Q}).\end{aligned}$$

Therefore the motion can be considered as a translation of Q (with *different* velocity that the center of mass) with rotation about Q (with the *same* angular velocity $\boldsymbol{\omega}$).

7.4.1 Equations of motion

As shown previously, the linear and angular momenta evolve according to

$$\dot{\mathbf{P}} = \mathbf{F} \quad \text{(total external force)} \quad \& \quad \dot{\mathbf{L}} = \mathbf{G} \quad \text{(total external torque)}$$

These two equations determine the translational and rotational motion of a rigid body.

$\mathbf{L}$ and $\mathbf{G}$ depend on the choice of origin, which could be any point that is fixed in an inertial frame. More surprisingly, these two equations can also be applied to the center of mass, even if this is accelerated:

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i, \Rightarrow m_i\ddot{\mathbf{r}}_i^c = \mathbf{F}_i + m_i\ddot{\mathbf{R}}.$$

So there is a fictitious force $m_i\ddot{\mathbf{R}}$ in the center-of-mass frame. But the total torque of the fictitious forces about the center of mass is

$$\sum_i \mathbf{r}_i^c \times \left(-m_i\ddot{\mathbf{R}}\right) = -\left(\sum m_i\mathbf{r}_i^c\right) \times \ddot{\mathbf{R}} = 0 \times \dot{\mathbf{R}} = 0.$$

In summary, the laws of motion apply in any inertial frame, or the center of mass in any frame.

7.4.2 Motion in a uniform gravitational field

In a uniform gravitational field $\mathbf{g}$, the total gravitational force and torque are the same as those that would act on a single particle of mass M located at the center of mass (which is also the *center of gravity*).

In particular, the gravitational torque about the center of mass vanishes: $\mathbf{G}^c = \mathbf{0}$.

We obtain a similar result for gravitational potential energy.

The gravitational potential in a uniform $\mathbf{g}$ is

$$\varphi_g = -\mathbf{r} \cdot \mathbf{g}.$$

(since $\mathbf{g} = -\nabla \phi_g$ by definition)

So

$$V^{\text{ext}} = \sum_i V_i^{\text{ext}} = \sum_i m_i(-\mathbf{r}_i \cdot \mathbf{g}) = M(-\mathbf{R} \cdot \mathbf{g}).$$

Example (Thrown stick)**.** Suppose we throw a symmetrical stick. So the center of mass is the actual center. Then the center of the stick moves in a parabola. Meanwhile, the stick rotates with constant angular velocity about its center due to the absence of torque.

7.4.3 Sliding versus rolling

Consider a cylinder or sphere of radius a, moving along a stationary horizontal surface.

In general, the motion consists of a translation of the center of mass (with velocity v) plus a rotation about the center of mass (with angular velocity ω).

The horizontal velocity at the point of contact is

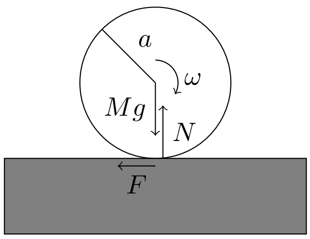
$$v_{\text{slip}} = v - a\omega.$$

For a pure sliding motion, $v \neq 0$ and $\omega = 0$, in which case $v - a\omega \neq 0$: the point of contact moves relative to the surface and kinetic friction may occur.

For a pure rolling motion, $v \neq 0$ and $\omega \neq 0$ such that $v - a\omega = 0$: the point of contact is stationary. This is the no-slip condition.

The rolling body can alternatively be considered to be rotating instantaneously about the point of contact (with angular velocity ω) and not translating.

Example (Snooker ball)**.**



It is struck centrally so as to initiate translation, but not rotation. Sliding occurs initially. Intuitively, we think it will start to roll, and we'll see that's the case.

The constant frictional force is

$$F = \mu_k N = \mu_k M g,$$

which applies while $v - a\omega > 0$.

The moment of inertia about the center of mass is

$$I = \frac{2}{5} M a^2.$$

The equations of motion are

$$M\dot{v} = -F$$
$$I\dot{\omega} = aF$$

Initially, $v = v_0$ and $\omega = 0$. Then the solution is

$$v = v_0 - \mu_k g t$$
$$\omega = \frac{5}{2}\frac{\mu_k g}{a}t$$

as long as $v - a\omega > 0$. The slip velocity is

$$v_{\text{slip}} = v - a\omega = v_0 - \frac{7}{2}\mu_k g t = v_0\left(1 - \frac{t}{t_{\text{roll}}}\right),$$

where

$$t_{\text{roll}} = \frac{2v_0}{7\mu_k g}.$$

This is valid up till $t = t_{\text{roll}}$. Then the slip velocity is 0, rolling begins and friction ceases.

At this point, $v = a\omega = \frac{6}{7}v_0$. The energy is then $\frac{5}{14}Mv_0^2 < \frac{1}{2}Mv_0^2$. So energy is lost to friction.

8 Special Relativity

When particles move *really* fast, Newtonian Dynamics becomes inaccurate and is replaced by Einstein's Special Theory of Relativity (1905).

Its effects are noticeable only when particles approach to the speed of light,

$$c = 299792458ms^{-1} \approx 3 * 10^8 ms^{-1}$$

The Special Theory of Relativity rests on the following postulate:

The laws of physics are the same in all inertial frames

This is the principle of relativity familiar to Galileo. Galilean relativity mentioned in the first chapter satisfies this postulate for dynamics. People then thought that Galilean relativity is what the world obeys. However, it turns out that there is a whole family of solutions that satisfy the postulate (for dynamics), and Galilean relativity is just one of them.

This is not a problem (yet), since Galilean relativity seems so intuitive, and we might as well take it to be the true one. However, it turns out that solving Maxwell's equations of electromagnetism gives an explicit value of the speed of light, c. This is independent of the frame of reference. So the speed of light must be the same in every inertial frame.

This is not compatible with Galilean relativity. Therefore, we need to find a different solution to the principle of relativity that preserves the speed of light.

8.1 The Lorentz transformation

Consider again inertial frames S and S' whose origins coincide at $t = t' = 0$. For now, neglect the y and z directions, and consider the relationship between (x, t) and (x', t'). The general form is

$$x' = f(x,t), \quad t' = g(x,t),$$

for some functions f and g. This is not very helpful.

In any inertial frame, a free particle moves with constant velocity. So straight lines in (x, t) must map into straight lines in (x', t'). Therefore the relationship must be *linear*.

Given that the origins of S and S' coincide at $t = t' = 0$, and S' moves with relativity v relative to S, we know that the line $x = vt$ must map into $x' = 0$.

Combining these two information, the transformation must be of the form

$$x' = \gamma(x - vt), \tag{1}$$

for some factor γ that may depend on $|v|$ (*not* v itself. We can use symmetry arguments to show that γ should take the same value for velocities v and $-v$).

Now reverse the roles of the frames. From the perspective S', S moves with velocity $-v$. A similar argument leads to

$$x = \gamma(x' + vt'), \tag{2}$$

with the same factor γ, since γ only depends on $|v|$. Now we know that light rays travel at the same speed in all frames:

$$x = ct \quad \& \quad x' = ct'$$

as well, so that the speed of light is the same in each frame. Substitute these into (1) and (2)

$$\begin{aligned} ct' &= \gamma(c - v)t \\ ct &= \gamma(c + v)t' \end{aligned}$$

Multiply the two equations together and divide by tt' to obtain

$$c^2 = \gamma^2(c^2 - v^2).$$

So

$$\gamma = \sqrt{\frac{c^2}{c^2 - v^2}} = \frac{1}{\sqrt{1 - (v/c)^2}}.$$

Definition (Lorentz factor)**.** The *Lorentz factor* is

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}}.$$

We still have to solve for the relation between t and t'. Eliminate x between (1) and (2) to obtain

$$x = \gamma(\gamma(x - vt) + vt').$$

So

$$t' = \gamma t - (1 - \gamma^{-2})\frac{\gamma x}{v} = \gamma\left(t - \frac{v}{c^2}x\right).$$

So we have

Law (Principle of Special Relativity)**.** Let S and S' be inertial frames, moving at the relative velocity of v. Then

$$x' = \gamma(x - vt)$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right),$$

where

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}}.$$

This is the *Lorentz transformations* in the standard configuration (in one spatial dimension).

We can invert this linear mapping to find (after some algebra)

$$x = \gamma(x' + vt')$$
$$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

Directions perpendicular to the relative motion of the frames are unaffected:

$$y' = y$$
$$z' = z$$

Now you can check that the speed of light is really invariant.

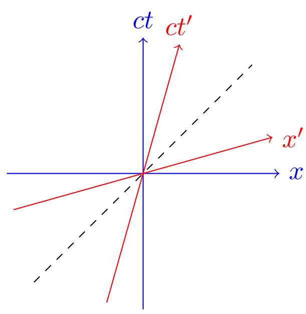
8.2 Spacetime Diagrams

When considering one spatial dimension (x) and time (t) in an inertial frame S. We plot x on the horizontal axis and ct on the vertical axis.

Definition. The union of space and time in special relativity is called *Minkowski spacetime*. Each point P represents an *event*, labelled by coordinates (ct, x).

A particle traces out a *world line* in spacetime, which is straight if the particle moves uniformly. Light rays moving in the x direction have world lines inclined at $45\circ$. We will see later that particles cannot move at speeds $|v| > c$.

We can also draw the axes of S', moving in the x direction at velocity v relative to S. The ct' axis corresponds to $x' = 0$, as in $x = \frac{v}{c}ct$. The x' axis corresponds to $t' = 0$, as in $ct = \frac{v}{c}x$.



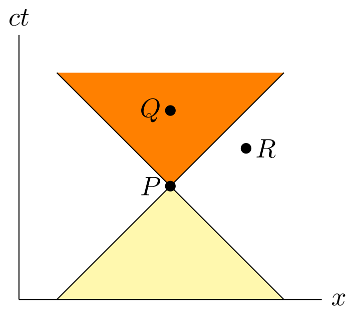
8.3 Relativistic Physics

Two events P_1 and P_2 are simultaneous in the frame S if $t_1 = t_2$. However, the lines of simultaneity in S and S' are not the same (obvious from axis), so events that happen simultaneously in one reference frame could happen at different times in another reference frame. Therefor simultaneity is relative.

8.3.1 Causality

Although differently moving observers may disagree on the temporal ordering of events, the consistent ordering of cause and effect can be ensured.

Lines of Simutaneity cannot be inclined at more than 45 degrees because Lorentz boosts are only possible for $|v| < cz$. All observers agree that Q occurs after P but different observers disagree on the temporal ordering of R. However, nothing travels faster than light , so P and R does not affect each other.



8.3.2 Time dilation

Suppose we have a clock that is stationary in S' (which travels at constant velocity v with respect to inertial frame S) ticks at constant intervals $\Delta t'$. What is the interval between ticks in S?

Lorentz transformation gives

$$t = \gamma \left(t' + \frac{v}{c^2} x' \right).$$

Since $x' =$ constant for the clock, we see that moving clocks go "slower":

$$\Delta t = \gamma \Delta t' > \Delta t'.$$

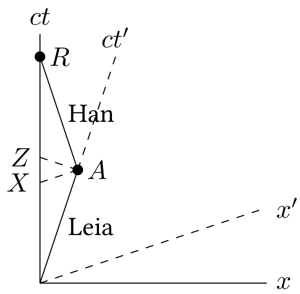
8.3.3 The twin paradox

Consider two twins: Luke and Leia. Luke stays at home. Leia travels at a constant speed v to a distant planet P, turns around, and returns at the same speed.

From Luke's perspective, by Leia's return R, Luke has aged by $2T$, but Leia has aged by $\frac{2T}{\gamma} < 2T$. So she is younger than Luke, because of time dilation.

The paradox is: From Leia's perspective, Luke travelled away from her at speed and the returned, so he should be younger than her!

Why is the problem not symmetric? Let's see in luke's frame. (Leia's in dashed lines)



In Leia's frame, by the time she arrives at A, she has experienced a time $T' = \frac{T}{\gamma}$. This event is simultaneous with event X in Leia's frame. Then in Luke's frame, the coordinates of X are

$$(ct, x) = \left(\frac{T'}{\gamma}, 0\right) = \left(\frac{T}{\gamma^2}, 0\right),$$

obtained through calculations similar to that above. So Leia thinks Luke has aged less by a factor of $1/\gamma^2$. At this stage, the problem *is* symmetric, and Luke also thinks Leia has aged less by a factor of $1/\gamma^2$.

Things change when Leia turns around and changes frame of reference. To understand this better, suppose Leia meets a friend, Han, who is just leaving A at speed v. Han also thinks Luke ages T/γ^2. But in his frame of reference, his departure is simultaneous with Luke's event Z, not X, since he has different lines of simultaneity.

So the asymmetry between Luke and Leia occurs when Leia turns around. At this point, she sees Luke age rapidly from X to Z.

8.3.4 Length contraction

A rod of length L' is stationary in S'. It's length in S is $L = L'/\gamma < L'$ by using the spacetime diagram.

Definition (Proper length)**.** The *proper length* is the length measured in an object's rest frame.

This is analogous to the fact that if you view a bar from an angle, it looks shorter than if you view it from the front. In relativity, what causes the contraction is not a spacial rotation, but a spacetime *hyperbolic* rotation.

8.3.5 Composition of Velocities

A particle moves with constant velocity u' in frame S', which moves with velocity v relative to S. What is its velocity u in S?

The world line of the particle in S' is $x' = u't'$. In S, using inverse Lorentz transformation: $u = \frac{x}{t} = \frac{x'+vt'}{t'+\frac{v}{c^2}x'} = \frac{u't'+vt'}{t'+\frac{v}{c^2}u't'} = \frac{u'+v}{1+\frac{u'v}{c^2}}$. This is the formula for the relativistic composition of velocities.

Remark. (i) When $u'v << c^2$, it reduces to the standard Galilean addition of velocities.

(ii) For given v ($|v| < c$ always), u is a monotonically, increasing function of u'.

(iii) When $u' = \pm c$, then $u = u'$ for any v.

(iv) When $|u'| < c$, $|u| < c$.

8.4 Geometry of Spacetime

8.4.1 The invariant interval

Consider events P and Q with coordinates (ct_1, x_1) and (ct_2, x_2), separate by $\delta t = t_2 - t_1$, $\delta x = x_2 - x_1$.

Definition. The *invariant interval* between P and Q is defined as $\delta s^2 = c^2 \delta t^2 - \delta x^2$.

We can check that all observers agree on the value of δs^2. In three spatial dimensions, we have $\delta s^2 = c^2 \delta t^2 - \delta x^2 - \delta y^2 - \delta z^2$. For two infinitesimally separated events, we have the *line element* $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$.

Spacetime is topologically equivalent to $\mathbb{R}^4$. When endowed with the distance measure δs^2 (which is not positive definite), it is called *Minkowski spacetime.* We say it has dimension $d = 1 + 3$.

Events with $\delta s^2 > 0$ are *timeline separated.* Events with $\delta s^2 < 0$ are *spacelike separated.* Events with $\delta s^2 = 0$ are *lightlike* or *null separated.*

8.5 The Lorentz Group

The coordinates of an event P in frame S can be written as a *4-vector.* (4-component vector) X.

$$X^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

The invariant interval between the origin and p can be written as an inner product.

$$X \cdot X = X^\mu \eta_{\mu\nu} X^\nu$$

where $\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is the *Minkowski Metric.* We see that $X \cdot X = x^2 t^2 - x^2 - y^2 - z^2$ as required So $X \cdot X > 0$ are timeline vectors, $X \cdot X < 0$ are spacelike, and $X \cdot X = 0$ are lightlike.

A lorentz transformation is a linear transformation of the coordinates from one frame (S) to another (S'), represented by a 4×4 matrix:

$$X' = \Lambda X$$

Lorentz transformations can be defined as those that leave the inner product invariant:

$$X' \cdot X' = X \cdot X$$

which implies the matrix equation:

$$\Lambda^T \eta \Lambda = \eta$$

Two classes of solution to this equation include:

- A 3*3 orthogonal matrix with an additional column and row on the top left (1 on diagonal, 0 everywhere else) that represent spatial rotations and reflections.

- $\begin{pmatrix} = -\gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ Where $\beta = \frac{v}{c}$, and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. These are Lorentz boosts in the x direction.

The set of all matrices of lorentz transformations form the Lorentz group O(1,3). It is generated by rotations and boosts, and includes spatial reflections and time reversals. The subgroup of $\det \eta = \pm 1$ is the *proper Lorentz group* SO(1,3). the subgroup that preserves spatial orientation and the direction of time is the *restricted lorentz group* $SO^{+}(1,3)$.

8.5.1 Rapidity

Focus on the upper left 2×2 matrix of Lorentz boosts in the x direction. Write:

$$\Lambda[\beta] = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$$

where $\gamma = \frac{1}{\sqrt{(1-\beta?2}}$ Combining two boosts int he x direction, we have:

$$\Lambda[\beta_1]\Lambda[\beta_2] = \begin{pmatrix} \gamma_1 & -\gamma_1\beta_1 \\ -\gamma_1\beta_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} \gamma_2 & -\gamma_2\beta_2 \\ -\gamma_2\beta_2 & \gamma_2 \end{pmatrix} = \Gamma[\frac{\beta_1\beta_2}{1+\beta_1\beta_2}$$

Tis is just the velocity composition formula as before. Recall,that, for spatial rotations:

$$R[\theta] = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$$

For lorentz boosts, we define the *rapidity* ϕ such that: $\beta = \tanh\phi$, $\gamma = \cosh\phi$, and $\gamma\beta = \sinh\phi$. Then

$$\Lambda[\beta] = \begin{pmatrix} \cosh x & -\sinh x \\ -\sinh x & \cosh x \end{pmatrix} = \Gamma(\phi)$$

And the rapidities add like rotation angles:

$$\Lambda(\phi_1)\Lambda(\phi_2) = \Lambda(\phi_1 + \phi_2)$$

This shows the close relationship between rotations and boosts. A boost is a *hyperbolic rotation* in spacetime.

8.6 Relativistic Kinematics

A particle moves along a trajectory $\mathbf{x}(t)$ in S. Its velocity in $\mathbf{u}(t) = \frac{d\mathbf{x}}{dt}$. However, there is a better description of the trajectory.

8.6.1 Proper Time

First consider a particle at rest in S', with $\mathbf{x}' = \mathbf{0}$. The invariant interval between events on its world line is:

$$\delta s^2 = c^2 \delta t^2$$

Definition. Define *proper time* τ such that:

$$\delta\tau = \frac{\delta s}{c}$$

τ is the time experienced by the particle. But the equation above holds in all frames, since δs is a Lorentz invariant. The world line of a particle can be parametrized using the proper time: $\mathbf{x}(\tau)$, $t(\tau)$.

Infinitesimal changes are related by

$$d\tau = \frac{ds}{c} = \sqrt{1 - \frac{|\mathbf{u}|^2}{c^2}}\mathrm{d}t$$

Thus $\frac{dt}{d\tau} = \gamma_u$ with $\gamma_u = \frac{1}{\sqrt{1-\frac{|u|^2}{c^2}}}$.

The total time experienced by the particle along a segment of its world line is:

$$T = \int d\tau = \int \frac{\mathrm{d}t}{\gamma_u}$$

8.6.2 4-Velocity

Definition. The *position 4-vector* of a particle is:

$$X(\tau) = \begin{pmatrix} ct(\tau) \\ \mathbf{x}(\tau) \end{pmatrix}$$

Its *4-velocity* is defined as:

$$U = \frac{dX}{dt} = \gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}$$

where $u = \frac{d\mathbf{x}}{dt}$.

If frame S and S' are related by $X' = \Lambda X$, then the 4-velocity transforms similarly:

$$U' = \Lambda U$$

Any 4-component vector that transforms in this way under a lorentz transformation is called a *4-vector.* U is a 4-vector because X is a 4-vector and τ is a Lorentz invariant. Note that $\frac{dX}{dt}$ is *not* a 4-vector.

The inner product is a Lorentz invariant, the same in all inertial frames. In the rest frame of the particle:

$$U \cdot U = c^2$$

We can check that this is also true in any other frame.

8.6.3 Transformation of velocities revisited

We have seen that velocity cannot be simply added in relativity. However, the 4-velocity does transform linearly according to the LT:

$$U' = \Lambda U$$

In frame S, consider a particle moving with speed U at angle θ to the x axis in the xy plane. Its 4-velocity is:

$$U = \begin{pmatrix} \gamma_u c \\ \gamma_u \cos\theta \\ \gamma_u \sin\theta \\ 0 \end{pmatrix}$$

Withe S and S' in standard configuration, we have:

$$\begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} \cos\theta' \\ \gamma_{u'} \sin\theta' \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_v & \frac{-\gamma_v v}{c} & 0 & 0 \\ -\frac{-\gamma_v v}{c} & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_u c \\ \gamma_u \cos\theta \\ \gamma_u \sin\theta \\ 0 \end{pmatrix}$$

The ratio of the second and first lines gives:

$$u' \cos\theta' = \frac{u\cos\theta - v}{1 - \frac{uv}{c?2}\cos\theta}$$

The third and second lines gives:

$$\tan\theta' = \frac{u\sin\theta}{\gamma_v(u\cos\theta - v)}$$

which describes aberration: a change in the apparent direction of motion of a particle due to the motion of the observer. Aberration of starlight due to Earth's orbital motion causes small annual changes in apparent positions of starts.

8.6.4 4-momentum

Definition (4-momentum)**.** The *4-momentum* of a particle of mass m is

$$P = mU = m\gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}$$

The 4-momentum of a system of particles is the sum of the 4-momentum of the particles, and is conserved in the absence of external forces.

The spatial components of P are the *relativistic 3-momentum*,

$$\mathbf{p} = m\gamma_u \mathbf{u},$$

which differs from the Newtonian expression by a factor of γ_u.

What is the interpretation of the time component P^0 (ie. the first time component of the P vector)? We expand for $|\mathbf{u}| \ll c$:

$$P^0 = m\gamma c = \frac{mc}{\sqrt{1 + |\mathbf{u}|^2/c^2}} = \frac{1}{c}\left(mc^2 + \frac{1}{2}m|\mathbf{u}|^2 + \cdots\right).$$

We have a constant term mc^2 plus a kinetic energy term $\frac{1}{2}m|\mathbf{u}|^2$, plus more tiny terms, all divided by c. So this suggests that P^0 is indeed the energy for a particle, and the remaining $\cdots$ terms are relativistic corrections for our old formula $\frac{1}{2}m|\mathbf{u}|^2$ (the mc^2 term will be explained later). So we interpret P as

$$P = \begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix}$$

Definition (Relativistic energy)**.** The *relativistic energy* of a particle is $E = P^0$. So

$$E = m\gamma c^2 = mc^2 + \frac{1}{2}m|\mathbf{u}|^2 + \cdots$$

Note that $E \to \infty$ as $|\mathbf{u}| \to c$.

For a stationary particle,

$$E = mc^2.$$

This implies that mass is a form of energy. m is sometimes called the *rest mass*.

The energy of a moving particle, $m\gamma_u c^2$, is the sum of the rest energy mc^2 and kinetic energy $m(\gamma_u - 1)c^2$.

Since $P \cdot P = \frac{E^2}{c^2} - |\mathbf{p}|^2$ is a Lorentz invariant (lengths of 4-vectors are always Lorentz invariant) and equals m^2c^2 in the particle's rest frame, we have the general relation between energy and momentum

$$E^2 = |\mathbf{p}|^2c^2 + m^2c^4$$

In Newtonian physics, mass and energy are separately conserved. In relativity, mass is not conserved. Instead, it is just another form of energy, and the total energy, including mass energy, is conserved.

Mass can be converged into kinetic energy and vice versa (eg. atomic bombs!)

Massless particles

Particles with zero mass ($m = 0$), eg. photons, can have non-zero momentum and energy because they travel at the speed of light ($\gamma = \infty$).

In this case, $P \cdot P = 0$. So massless particles have light-like (or null) trajectories, and no proper time can be defined for such particles.

For these particles, energy and momentum are related by

$$E^2 = |\mathbf{p}|^2c^2. \Rightarrow E = |\mathbf{p}|c. \Rightarrow P = \frac{E}{c}\begin{pmatrix} 1 \\ \mathbf{n} \end{pmatrix},$$

where $\mathbf{n}$ is a unit (3-)vector in the direction of propagation.

According to quantum mechanics, fundamental "particles" aren't really particles but have both particle-like and wave-like properties (if that sounds confusing, yes it is!). Hence we can assign it a *de Broglie wavelength*, according to the *de Broglie relation*:

$$|\mathbf{p}| = \frac{h}{\lambda}$$

where $h \approx 6.63 * 10^{-34}\ \ m^2kg\,s^{-1}$ is *Plank's constant*.

For massless particles, this is consistent with *Planck's relation*:

$$E = \frac{hc}{\lambda} = h\nu,$$

where $\nu = \frac{c}{\lambda}$ is the *wave frequency*.

Newton's second law in special relativity

Definition (4-force)**.** The *4-force* is

$$F = \frac{\mathrm{d}P}{\mathrm{d}\tau}$$

This equation is the relativistic counterpart to Newton's second law.
It is related to the 3-force $\mathbf{F}$ by

$$F = \gamma_u \begin{pmatrix} \mathbf{F} \cdot \mathbf{u}/c \\ \mathbf{F} \end{pmatrix}$$

Expanding the definition of the 4-force componentwise, we obtain

$$\frac{\mathrm{d}E}{\mathrm{d}\tau} = \gamma_u \mathbf{F} \cdot \mathbf{u} \Rightarrow \frac{\mathrm{d}E}{\mathrm{d}t} = \mathbf{F} \cdot \mathbf{u}$$

and

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = \gamma_u \mathbf{F} \Rightarrow \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}$$

Equivalently, for a particle of mass m,

$$F = mA,$$

where

$$A = \frac{\mathrm{d}U}{\mathrm{d}\tau}$$

is the 4-acceleration.

We have

$$U = \gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}$$

So

$$A = \gamma_u \frac{\mathrm{d}U}{\mathrm{d}t} = \gamma_u \begin{pmatrix} \dot{\gamma}_u c \\ \gamma_u \mathbf{a} + \dot{\gamma}_u \mathbf{u}. \end{pmatrix}$$

where $\mathbf{a} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}$ and $\dot{\gamma}_u = \gamma_u^3 \frac{\mathbf{a} \cdot \mathbf{u}}{c^3}$.

In the instantaneous rest frame of a particle, $\mathbf{u} = \mathbf{0}$ and $\gamma_u = 1$. So

$$U = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}, \quad A = \begin{pmatrix} 0 \\ \mathbf{a} \end{pmatrix}$$

Then $\mathbf{U} \cdot \mathbf{A} = 0$. Since this is a Lorentz invariant, we have $\mathbf{U} \cdot \mathbf{A} = 0$ in all frames.

8.7 Particle Physics

Many problems can be solved using the conservation of 4-momentum.

$$P = \begin{pmatrix} \frac{E}{c} \\ \mathbf{p} \end{pmatrix}$$

for a system of particles. The *centre-of-momentum* frame is an inertial frame in which the total 3-momenum $\mathbf{p} = \mathbf{0}$.

8.7.1 Particle Decay

A particle of mass m_1 decays into two particles of masses m_2 and m_3. We have $p_1 = p_2 + p_3$, so $E_1 = E_2 + E_3$, and $\mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_3$ in the CM frame.

$$E_1 = \sqrt{|p_2|^2c^2 + m_2^2c^4} + \sqrt{|p_3|^2c^2 + m_3^2c^4} \geq m + m_2c^2 + m_3c^2$$

so decay is possible only if $m_1 \geq m_2 + m_3$.

Remark. Mass is *NOT* conserved in relativity!

Example.

$$h \to \gamma\gamma$$

is possible (higgs to 2 photons) by criterion above, because $m_h > 0$ and $m_\gamma = 0$. In the Higgs' rest frame, we have $\mathbf{p}_{\gamma_1} = -\mathbf{p}_{\gamma_2}$ so $\mathbf{E}_{\gamma_1} = \mathbf{E}_{\gamma_2} = \frac{1}{2}m_hc^2$.

8.7.2 Particle Scattering

When two particles collide and retain their identities, the total 4-momentum is conserved:

$$P_1 + P_2 = P_3 + P_4$$

In the laboratory frame S, suppose particle 1 travels with speed u and collides with particle 2 (at rest).

In the CM frame S',:

$$\mathbf{p}_1' + \mathbf{p}_2' = \mathbf{0} = \mathbf{p}_3' + \mathbf{p}_4'$$

both before and after the collisions, the two particles have equal and opposite 4-momentum. The scattering angle itself is undetermined due to quantum mechanics.

Suppose the particles have equal mass M. Then they have the same speed v in S'. Choose axes such that:

$$P_1' = \begin{pmatrix} m\gamma_v c \\ m\gamma_v v \\ 0 \\ 0 \end{pmatrix}, P_2' = \begin{pmatrix} m\gamma_v c \\ -m\gamma_v v \\ 0 \\ 0 \end{pmatrix}, P_3' = \begin{pmatrix} m\gamma_v c \\ m\gamma_v v\cos\theta' \\ m\gamma_v v\sin\theta' \\ 0 \end{pmatrix}, P_4' = \begin{pmatrix} m\gamma_v c \\ -m\gamma_v v\cos\theta' \\ -m\gamma_v v\sin\theta' \\ 0 \end{pmatrix}$$

Transform back to the laboratory frame S, in which:

$$P_1 = \begin{pmatrix} m\gamma_u c \\ m\gamma_u u \\ 0 \\ 0 \end{pmatrix}, P_2 = \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Using the inverse Lorentz transform matrix.

We find that :

$$u = \frac{2v}{1 + \frac{v^2}{c^2}}$$

$$\tan\theta = \frac{\sin\theta'}{\gamma_v(1+\cos\theta')} = \frac{1}{\gamma_v}\tan(\frac{\theta'}{2})$$

$$\tan\phi = \frac{\sin\theta'}{\gamma_v(1-\cos\theta')} = \frac{1}{\gamma_v}\cot(\frac{\theta'}{2})$$

Thus $\tan\theta\tan\phi = \frac{1}{\gamma_v^2}$.

In the newtonian limit, the outgoing trajectories are perpendicular in S.

8.7.3 Particle creation

Collide two particles of mass m fast enough to create an extra particle of mass M.

$$P_1 + P_2 = P_3 + P_4 + P_5$$

In the CM frame:

$$P_1 + P_2 = \begin{pmatrix} 2m\gamma_v c \\ 0 \end{pmatrix}$$

$$P_3 + P_4 + P_5 = \begin{pmatrix} \frac{E_3+E_4+E_5}{c} \\ 0 \end{pmatrix}$$

So $2m\gamma_v c^2 = E_3 + E_4 + E_5 \geq 2mc^2 + Mc^2 \Rightarrow \gamma_v \geq 1 + \frac{M}{2m}$.

This means that the initial kinetic energy in the CM frame must be

$$2(\gamma_v - 1)mc^2 \geq Mc^2$$

[missing part]

Transform to a frame in which the initial speeds are u and 0, then :

$$u = \frac{2v}{1 + \frac{v^2}{c^2}}$$

$$\gamma_u = 2\gamma_v^2 - 1 \Rightarrow \gamma_u \geq 1 + \frac{2M}{m} + \frac{M^2}{2m^2}$$

This means that the initial kinetic energy in this frame must be

$$m(\gamma_u - 1)c^2 = \left(2 + \frac{M}{2m}\right) Mc^2,$$

which could be much larger than Mc^2, especially if $M \gg m$, which usually the case. For example, the mass of the Higgs boson is 130 times the mass of the proton. So it would be much advantageous to collide two beams of protons head on, as opposed to hitting a fixed target.