Complex Analysis Review Sheet

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Analytic functions
Complex differentiation and the Cauchy-Riemann equations. Examples. Conformal mappings. Informal discussion of branch points, examples of $\log z$ and z^c. [3]

Contour integration and Cauchy's theorem
Contour integration (for piecewise continuously differentiable curves). Statement and proof of Cauchy's theorem for star domains. Cauchy's integral formula, maximum modulus theorem, Liouville's theorem, fundamental theorem of algebra. Morera's theorem. [5]

Expansions and singularities
Uniform convergence of analytic functions; local uniform convergence. Differentiability of a power series. Taylor and Laurent expansions. Principle of isolated zeros. Residue at an isolated singularity. Classification of isolated singularities. [4]

The residue theorem
Winding numbers. Residue theorem. Jordan's lemma. Evaluation of definite integrals by contour integration. Rouché's theorem, principle of the argument. Open mapping theorem. [4]

Contents

1 Complex Differentiation

Now, if you treat i just to be another dimension (which is *just about* correct), then we are essentially differentiating functions that go from $\mathbb{R}^{\nvDash} \to \mathbb{R}^2$. Following is a list of (not very useful) but needed definitions:

Definition (Open subset)**.** A subset $U \subseteq \mathbb{C}$ is *open* if for any $x \in U$, there is some $\varepsilon > 0$ such that the open ball $B_\varepsilon(x) = B(x; \varepsilon) \subseteq U$.

Definition (Path-connected subset)**.** A subset $U \subseteq \mathbb{C}$ is path-connected if for any $x, y \in U$, there is some $\gamma : [0, 1] \to U$ continuous such that $\gamma(0) = x$ and $\gamma(1) = y$. We call a non-empty open path-connected subset of $\mathbb{C}$ to be a domain.

Definition (Differentiable function)**.** Let $U \subseteq \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ be a function. We say f is *differentiable* at $w \in U$ if

$$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w}$$

exists.

Now as we defined differentiability, we can define some set of functions that are very nice:

Definition (Analytic/holomorphic function)**.** A function f is *analytic* or *holomorphic* at $w \in U$ if f is differentiable on an open neighbourhood $B(w, \varepsilon)$ of w (for some ε).

Definition (Entire function)**.** If $f : \mathbb{C} \to \mathbb{C}$ is defined on all of $\mathbb{C}$ and is holomorphic on $\mathbb{C}$, then f is said to be *entire*.

Now it turns out that entire functions are not *that* interesting, because we (will) know that they are just polynomials and (some) power series. We are more interested on those defined in a subset of $\mathbb{C}$ and we want to see if they are differentiable. As we noted earlier, this is basically just differentiating in $\mathbb{R}^2 \to \mathbb{R}^2$, so we have the following:

Proposition (Cauchy-Riemann Equations)**.** Let f be defined on an open set $U \subseteq \mathbb{C}$. Let $w = c + id \in U$ and write $f = u + iv$. Then f is complex differentiable at w if and only if u and v, viewed as a real function of two real variables, are differentiable at (c, d), *and*

$$u_x = v_y \qquad u_y = -v_x.$$

These equations are the *Cauchy-Riemann equations*. In this case, we have $f'(w) = u_x(c, d) + iv_x(c, d) = v_y(c, d) - iu_y(c, d)$.

Proof. By definition, f is differentiable at w with $f'(w) = p + iq$ if and only if

$$\lim_{z \to w} \frac{f(z) - f(w) - f'(w)(z - w)}{z - w} = 0. \tag{†}$$

If $z = x + iy$, then

$$f'(w)(z - w) = p(x - c) - q(y - d) + i(q(x - c) + p(y - c)).$$

So, breaking into real and imaginary parts, we know (†) holds if and only if

$$\lim_{(x,y)\to(c,d)} \frac{u(x,y) - u(c,d) - (p(x-c) - q(y-d))}{\sqrt{(x-c)^2 + (y-d)^2}} = 0$$

and

$$\lim_{(x,y)\to(c,d)} \frac{v(x,y) - v(c,d) - (q(x-c) + p(y-d))}{\sqrt{(x-c)^2 + (y-d)^2}} = 0.$$

Comparing this to the definition of the differentiability of a real-valued function, we see this holds exactly if u and v are differentiable at (c, d) with

$$Du|_{(c,d)} = (p, -q), \quad Dv|_{(c,d)} = (q, p).$$

□

1.1 Conformal Mappings

Note. This section has almost no relation to the other sections but it is in the syllabus. So we put it here.

We first define a few terms before we can understand what is it talking about:

Definition (Conformal function)**.** Let $f : U \to \mathbb{C}$ be a function holomorphic at $w \in U$. If $f'(w) \neq 0$, we say f is *conformal* at w.

Note. This implies two things:

- This function is locally invertible by the Inverse Function Theorem.
- Angles are preserved. If we define $\gamma_1, \gamma_2 : [-1, 1] \to U$ be continuously differentiable paths that intersect when $t = 0$ at $w = \gamma_1(0) = \gamma_2(0)$, and $\text{angle}(\gamma_1, \gamma_2) = \arg(\gamma_1'(0)) - \arg(\gamma_2'(0))$, then we have, for a conformal function f:

$$\begin{aligned}\text{angle}(f \circ \gamma_1, f \circ \gamma_2) &= \arg((f \circ \gamma_1)'(0)) - \arg((f \circ \gamma_2)'(0)) \\ &= \arg\left(\frac{(f \circ \gamma_1)'(0)}{(f \circ \gamma_2)'(0)}\right) = \arg\left(\frac{\gamma_1'(0)}{\gamma_2'(0)}\right) = \text{angle}(\gamma_1, \gamma_2),\end{aligned}$$

  So the angles are preserved (using the chain rule).

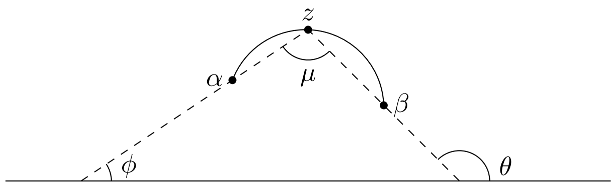
Definition (Conformal equivalence)**.** If U and V are open subsets of $\mathbb{C}$ and $f : U \to V$ is a conformal bijection, then it is a *conformal equivalence*.

Note. The main takeaway from this section is "how to create functions that map different regions conformally". Remember that angles are preserved! These are some commonly used tricks:

- Mobius Maps. This maps circles/lines to cicles/lines. Great for mapping quadrants (which have *line* boundaries) to circles and vice versa.

  They are also *great* at sending lunes, which are intersections of two circles to quadrants. Just map the two corners of the lunes to 0 and infinity, and you will have a straight line.

- Rotations. e^{ix}. You know what they do.
- Powers. The nth power sends smaller sectors ($0 < \arg(z) < \frac{\pi}{n}$) to larger ones ($0 < \arg(z) < \pi$)
- Circular arc transformations. Suppose we have a circular arc described by:



Then in a region bound by two curves, with the one above μ_- and one above μ_+, we can describe the region as:

$$\left\{z : \arg\left(\frac{z-\alpha}{z-\beta}\right) \in [\mu_-, \mu_+]\right\}.$$

Then we can map

$$z \mapsto \left(\frac{z-\alpha}{z+\beta}\right)^k$$

Gives a quadrant like object (for an appropriate power k).

- Exponentials. They map $\{z : \Re(z) \in (a, b)\}$ to the annulus $\{e^a \leq |z| \leq e^b\}$. Note this is *not* a bijection because we are mapping between simply connected regions.
- Logs. They map reversely of the exponentials, and similarly are not bijections.

We also have an important theorem that we wouldn't prove:

Definition (Simple closed curve)**.** A *simple closed curve* is the image of an injective map $S^1 \to \mathbb{C}$.

Definition (Simply connected)**.** A domain $\mathcal{U} \subseteq \mathbb{C}$ is *simply connected* if every continuous map from the circle $f : S^1 \to \mathcal{U}$ can be extended to a continuous map from the disk $F : \overline{D^2} \to \mathcal{U}$ such that $F|_{\partial \overline{D^2}} = f$. Alternatively, any loop can be continuously shrunk to a point.

Theorem (Riemann mapping theorem)**.** Let $\mathcal{U} \subseteq \mathbb{C}$ be the bounded domain enclosed by a simple closed curve, or more generally any simply connected domain not equal to all of $\mathbb{C}$. Then $\mathcal{U}$ is conformally equivalent to $D = \{z : |z| < 1\} \subseteq \mathbb{C}$.

1.2 Power Series

Now we start by redefining some terms and restating some theorems in Analysis II:

Definition (Uniform convergence)**.** A sequence (f_n) of functions *converge uniformly* to f if for all $\varepsilon > 0$, there is some N such that $n > N$ implies $|f_n(z) - f(z)| < \varepsilon$ for all z.

Proposition. The uniform limit of continuous functions is continuous.

Proposition (Weierstrass M-test)**.** For a sequence of functions f_n, if we can find $(M_n) \subseteq \mathbb{R}_{>0}$ such that $|f_n(x)| < M_n$ for all x in the domain, then $\sum M_n$ converges implies $\sum f_n(x)$ converges uniformly on the domain.

Proposition. Given any constants $\{c_n\}_{n\geq 0} \subseteq \mathbb{C}$, there is a unique $R \in [0, \infty]$ such that the series $z \mapsto \sum_{n=0}^{\infty} c_n(z-a)^n$ converges absolutely if $|z-a| < R$ and diverges if $|z-a| > R$. Moreover, if $0 < r < R$, then the series converges uniformly on $\{z : |z-a| < r\}$. This R is known as the *radius of convergence*.

Now we would prove an important theorem about power series:

Theorem. Let
$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$$
be a power series with radius of convergence $R > 0$. Then

(i) f is holomorphic on $B(a; R) = \{z : |z-a| < R\}$.

(ii) $f'(z) = \sum nc_n(z-1)^{n-1}$, which also has radius of convergence R.

(iii) Thus f is infinitely complex differentiable on $B(a; R)$ with $c_n = \frac{f^{(n)}(a)}{n!}$.

Proof. Without loss of generality, take $a = 0$.

Certainly, we have $|nc_n| \geq |c_n|$. So by comparison to the series for f, we can see that the radius of convergence of $\sum nc_n z^{n-1}$ is at most R. But if $|z| < \rho < R$, then we can see
$$\frac{|nc_n z^{n-1}|}{|c_n \rho^{n-1}|} = n\left|\frac{z}{\rho}\right|^{n-1} \to 0$$
as $n \to \infty$. So by comparison to $\sum c_n \rho^{n-1}$, which converges, we see that the radius of convergence of $\sum nc_n z^{n-1}$ is at least ρ. So the radius of convergence must be exactly R.

Now we want to show f really is differentiable with that derivative. Pick z, w such that $|z|, |w| \leq \rho$ for some $\rho < R$ as before.

Define a new function
$$\varphi(z, w) = \sum_{n=1}^{\infty} c_n \sum_{j=0}^{n-1} z^j w^{n-1-j}.$$

Noting
$$\left| c_n \sum_{j=0}^{n-1} z^j w^{n-1-j} \right| \leq n|c_n|\rho^n,$$
we know the series defining φ converges uniformly on $\{|z| \leq \rho, |w| < \rho\}$, and hence to a continuous limit.

If $z \neq w$, then using the formula for the (finite) geometric series, we know
$$\varphi(z, w) = \sum_{n=1}^{\infty} c_n \left(\frac{z^n - w^n}{z - w}\right) = \frac{f(z) - f(w)}{z - w}.$$

On the other hand, if $z = w$, then

$$\varphi(z, z) = \sum_{n=1}^{\infty} c_n n z^{n-1}.$$

Since φ is continuous, we know

$$\lim_{w \to z} \frac{f(z) - f(w)}{z - w} \to \sum_{n=1}^{\infty} c_n n z^{n-1}.$$

So $f'(z) = \varphi(z, z)$ as claimed. Then (iii) follows from (i) and (ii) directly. $\square$

1.3 Logarithms and Branch Cuts

Note. This is important. Like very important. Know your branch cuts!

Definition (Branch of logarithm)**.** Let $U \subseteq \mathbb{C}^*$ be an open subset. A *branch of the logarithm* on U is a continuous function $\lambda : U \to \mathbb{C}$ for which $e^{\lambda(z)} = z$ for all $z \in U$.

Note that we cannot have a branch of logarithm on the whole $\mathbb{C}^*$ (which we would prove later). We usually define it through cutting the negative axis and define the *principal branch*, which is that we write $z = r^{i\theta}$ with $\theta \in (-\pi, pi)$. Then we define $\log(z) = \log(r) + i\theta$. Now we prove that it has the property we want:

Proposition. On $\{z \in \mathbb{C} : z \notin \mathbb{R}_{\leq 0}\}$, the principal branch $\log : U \to \mathbb{C}$ is holomorphic function. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}z} \log z = \frac{1}{z}.$$

If $|z| < 1$, then

$$\log(1 + z) = \sum_{n \geq 1} (-1)^{n-1} \frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots .$$

Proof. That logarithm is holomorphic follows from the chain rule and $e^{\log z} = z$. This shows $\frac{\mathrm{d}}{\mathrm{d}z}(\log z) = \frac{1}{z}$.

To show that $\log(1 + z)$ is indeed given by the said power series, note that the power series does have a radius of convergence 1 by, say, the ratio test. So by the previous result, it has derivative

$$1 - z + z^2 + \cdots = \frac{1}{1 + z}.$$

Therefore, $\log(1 + z)$ and the claimed power series have equal derivative, and hence coincide up to a constant. Since they agree at $z = 0$, they must in fact be equal. $\square$

Then we can define powers using the logarithm, with $z^\alpha = e^{\alpha \log z}$. Thus, note that this is *ONLY* defined when log is. So when doing integrals, pay attention!

2 Complex Integration

2.1 Introduction and Basic Properties

Now if we just have a function $f : [a, b] \to \mathbb{C}$ it is obvious how to generalize the Riemann integration here (separate into real and imaginary parts). But what if we don't? Let's first define a few terms:

Definition (Path)**.** A *path* in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$, where $a, b \in \mathbb{R}$. We call it *simple* if $\gamma(t_1) = \gamma(t_2)$ implies $t_1 = t_2$ or $t_1, t_2 = \{a, b\}$. We call it *closed* if $\gamma(a) = \gamma(b)$. Also, a simple closed path which is piecewise continuously differentiable is a *contour*.

Now we define complex integration, which, just like integration in $\mathbb{R}^2$, needs to be done over a path:

Definition (Complex integration)**.** If $\gamma : [a, b] \to U \subseteq \mathbb{C}$ is C^1-smooth and $f : U \to \mathbb{C}$ is continuous, then we define the *integral* of f along γ as

$$\int_\gamma f(z)\ \mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\ \mathrm{d}t.$$

By summing over subdomains, the definition extends to piecewise C^1-smooth paths, and in particular contours.

This satisfies all the usual properties (addition, multiplication, substitution) of a normal integral. Then we define the Antiderivative and the Fundamental theorem of calculus:

Definition (Antiderivative)**.** Let $U \subseteq \mathbb{C}$ and $f : U \to \mathbb{C}$ be continuous. An *antiderivative* of f is a holomorphic function $F : U \to \mathbb{C}$ such that $F'(z) = f(z)$.

Theorem (Fundamental Theorem of Calculus (FTC))**.** Let $f : U \to \mathbb{C}$ be continuous with antiderivative F. If $\gamma : [a, b] \to U$ is piecewise C^1-smooth, then

$$\int_\gamma f(z)\ \mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

Now if we integrate $\frac{1}{z}$ along the unit circle, we have:

$$\int_0^{2\pi} e^{-i\theta} i e^{i\theta} \mathrm{d}\theta = 2\pi i$$

Which does not vanish. Therefore, we do not have a continuous antiderivative of $\log$ around $z = 0$, or otherwise, the integral would be 0 by FTC.

2.2 Cauchy's Theorem

Proposition. Let $U \subseteq \mathbb{C}$ be a domain (ie. path-connected non-empty open set), and $f : U \to \mathbb{C}$ be continuous. Moreover, suppose

$$\int_\gamma f(z)\ \mathrm{d}z = 0$$

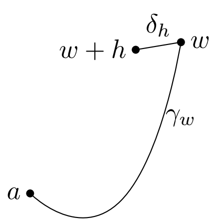
for any closed piecewise C^1-smooth path γ in U. Then f has an (holomorphic) antiderivative.

Proof. Pick our favorite $a_0 \in U$. For $w \in U$, we choose a path $\gamma_w : [0,1] \to U$ such that $\gamma_w(0) = a_0$ and $\gamma_w(1) = w$. Now this path is C^1 because we can take piecewise straight functions inside the union of balls around a continuous path γ, which we know it exists as this is a domain. We thus define

$$F(w) = \int_{\gamma_w} f(z)\,\mathrm{d}z.$$

Note. This is path independent, as for any other path from a_0 to w, we can form a contour around f with these paths, and since the difference of two integrals is 0 by the hypothesis, the two integrals agree.

Now we need to check that F is complex differentiable. Since U is open, we can pick $\theta > 0$ such that $B(w;\varepsilon) \subseteq U$. Let δ_h be the radial path in $B(w,\varepsilon)$ from W to $w+h$, with $|h| < \varepsilon$.



Now note that $\gamma_w * \delta_h$ is a path from a_0 to $w+h$. So

$$\begin{aligned} F(w+h) &= \int_{\gamma_w * \delta_h} f(z)\,\mathrm{d}z \\ &= F(w) + \int_{\delta_h} f(z)\,\mathrm{d}z \\ &= F(w) + hf(w) + \int_{\delta_h} (f(z) - f(w))\,\mathrm{d}z. \end{aligned}$$

Thus, we know

$$\begin{aligned} \left|\frac{F(w+h) - F(w)}{h} - f(w)\right| &\leq \frac{1}{|h|}\left|\int_{\delta_h} f(z) - f(w)\,\mathrm{d}z\right| \\ &\leq \frac{1}{|h|}|h| \sup_{\delta_h} |f(z) - f(w)| \\ &= \sup_{\delta_h} |f(z) - f(w)|. \end{aligned}$$

Since f is continuous, as $h \to 0$, we know $f(z) - f(w) \to 0$. So F is differentiable with derivative f. ☐

Note that we really didn't need to have the integral curve to be 0 in all possible curves. We just needed ones with straight lines. So we define:

Definition (Star-shaped domain)**.** A *star-shaped domain* or *star domain* is a domain U such that there is some $a_0 \in U$ such that the line segment $[a_0, w] \subseteq U$ for all $w \in U$.

Not this is *not* a convex domain! We only need 1 point that connects everywhere. Now to use the proof above, we needed that $\int_{\gamma_w * \delta_h} f(z)\,\mathrm{d}z = \int_{\gamma_{w+h}} f(z)\,\mathrm{d}z$. So we needed the integrals to vanish on a triangle:

Definition (Triangle)**.** A *triangle* in a domain U is what it ought to be — the Euclidean convex hull of 3 points in U, lying *wholly* in U. We write its boundary as ∂T, which we view as an oriented piecewise C^1 path, ie. a contour.

So we can restate our earlier theorem as:

Proposition. If U is a star domain, and $f : U \to \mathbb{C}$ is continuous, and if

$$\int_{\partial T} f(z)\,\mathrm{d}z = 0$$

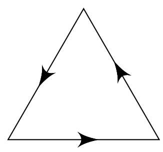
for all triangles $T \subseteq U$, then f has an antiderivative on U.

Now the following theorem is the more fundamental piece:

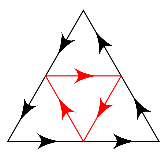
Theorem (Cauchy's theorem for a triangle)**.** Let U be a domain, and let $f : U \to \mathbb{C}$ be holomorphic. If $T \subseteq U$ is a triangle, then $\int_{\partial T} f(z)\,\mathrm{d}z = 0$.

By combining the two pieces above, we can see that integrals of holomorphic are 0 in any path in a star-shaped domain. Actually the theorem is true in any simply-connected region, but we would not prove it here.

Proof. Fix a triangle T. Let $\eta = \left|\int_{\partial T} f(z)\,\mathrm{d}z\right|$ and $\ell = \mathrm{length}(\partial T)$. The idea is to show to bound η by ε, for every $\varepsilon > 0$, and hence we must have $\eta = 0$. To do so, we subdivide our triangles. We start with $T = T^0$:



We then add more lines to get $T^0_a, T^0_b, T^0_c, T^0_d$.



We orient the middle triangle by the anti-clockwise direction. Then we have

$$\int_{\partial T^0} f(z)\,\mathrm{d}z = \sum_{a,b,c,d} \int_{\partial T^0_\cdot} f(z)\,\mathrm{d}z,$$

since each internal edge occurs twice, with opposite orientation.

For this to be possible, then there must be some subscript in $\{a, b, c, d\}$ such that

$$\left|\int_{\partial T^0_\cdot} f(z)\,\mathrm{d}z\right| \geq \frac{\eta}{4}.$$

We call this $T^0 = T^1$. Then we notice ∂T^1 has length

$$\text{length}(\partial T^1) = \frac{\ell}{2}.$$

Iterating this, we obtain triangles

$$T^0 \supseteq T^1 \supseteq T^2 \supseteq \cdots$$

such that

$$\left|\int_{\partial T^i} f(z)\, \mathrm{d}z\right| \geq \frac{\eta}{4^i}, \quad \text{length}(\partial T^i) = \frac{\ell}{2^i}.$$

Now we are given a nested sequence of closed sets, so there is some $z_0 \in \bigcap_{i \geq 0} T^i$.

Now fix an $\varepsilon > 0$. Since f is holomorphic at z_0, we can find a $\delta > 0$ such that

$$|f(w) - f(z_0) - (w - z_0)f'(z_0)| \leq \varepsilon |w - z_0|$$

whenever $|w - z_0| < \delta$. Since the diameters of the triangles are shrinking each time, we can pick an n such that $T^n \subseteq B(z_0, \varepsilon)$. We're almost there. Now 1 and z definitely have anti-derivatives on T^n. Therefore, noting that $f(z_0)$ and $f'(z_0)$ are just constants, we have

$$\begin{aligned}
\left|\int_{\partial T^n} f(z)\, \mathrm{d}z\right| &= \left|\int_{\partial T^n} (f(z) - f(z_0) - (z - z_0)f'(z_0))\, \mathrm{d}z\right| \\
&\leq \int_{\partial T^n} |f(z) - f(z_0) - (z - z_0)f'(z_0)|\, \mathrm{d}z \\
&\leq \text{length}(\partial T^n)\varepsilon \sup_{z \in \partial T^n} |z - z_0| \\
&\leq \varepsilon \text{length}(\partial T^n)^2,
\end{aligned}$$

where the last line comes from the fact that $z_0 \in T^n$, and the distance between any two points in the triangle cannot be greater than the perimeter of the triangle. Substituting our formulas for these in, we have

$$\frac{\eta}{4^n} \leq \frac{1}{4^n}\ell^2\varepsilon. \Rightarrow \eta \leq \ell^2 \varepsilon.$$

Since ℓ is fixed and ε was arbitrary, it follows that we must have $\eta = 0$. □

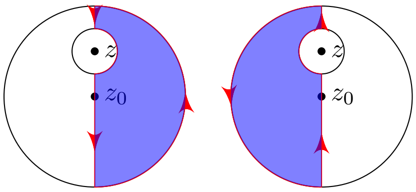
2.3 The Cauchy Integral Formula

Theorem (Cauchy integral formula)**.** Let U be a domain, and $f : U \to \mathbb{C}$ be holomorphic. Suppose there is some $\overline{B(z_0; r)} \subseteq U$ for some z_0 and $r > 0$. Then for all $z \in B(z_0; r)$, we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial \overline{B(z_0; r)}} \frac{f(w)}{w - z}\, \mathrm{d}w.$$

There are two proofs for this, but we would give the more graphical one as it is easier to apply on the test:

Proof. Given $\varepsilon > 0$, we pick $\delta > 0$ such that $\overline{B(z, \delta)} \subseteq B(z_0, r)$, and such that whenever $|w - z| < \delta$, then $|f(w) - f(z)| < \varepsilon$. This is possible since f is uniformly continuous on the neighbourhood of z. We now cut our region apart:



We know $\frac{f(w)}{w-z}$ is holomorphic on sufficiently small open neighbourhoods of the half-contours indicated. The area enclosed by the contours might not be star-shaped, but we can definitely divide it once more so that it is. Hence the integral of $\frac{f(w)}{w-z}$ around the half-contour vanishes by Cauchy's theorem. Adding these together, we get

$$\int_{\partial \overline{B(z_0,r)}} \frac{f(w)}{w-z}\,\mathrm{d}w = \int_{\partial \overline{B(z,\delta)}} \frac{f(w)}{w-z}\,\mathrm{d}w,$$

where the balls are both oriented anticlockwise. Now we have

$$\left| f(z) - \frac{1}{2\pi i}\int_{\partial \overline{B(z_0,r)}} \frac{f(w)}{w-z}\,\mathrm{d}w \right| = \left| f(z) - \frac{1}{2\pi i}\int_{\partial \overline{B(z,\delta)}} \frac{f(w)}{w-z}\,\mathrm{d}w \right|.$$

Now we once again use the fact that

$$\int_{\partial \overline{B(z,\delta)}} \frac{1}{w-z}\,\mathrm{d}z = 2\pi i$$

to show this is equal to

$$\left| \frac{1}{2\pi i}\int_{\partial \overline{B(z,\delta)}} \frac{f(z)-f(w)}{w-z}\,\mathrm{d}w \right| \leq \frac{1}{2\pi}\cdot 2\pi\delta\cdot\frac{1}{\delta}\cdot\varepsilon = \varepsilon.$$

Taking $\varepsilon \to 0$, we see that the Cauchy integral formula holds. □

Now we can prove that a non-constant function cannot achieve its maximum inside a region:

Corollary (Local maximum principle)**.** Let $f : B(z,r) \to \mathbb{C}$ be holomorphic. Suppose $|f(w)| \leq |f(z)|$ for all $w \in B(z;r)$. Then f is constant. In other words, a non-constant function cannot achieve an interior local maximum.

Proof. Let $0 < \rho < r$. We now set $w = z + \rho e^{2\pi i\theta}$. Applying the Cauchy integral formula, we get

$$|f(z)| = \left| \frac{1}{2\pi i}\int_{\partial \overline{B(z;\rho)}} \frac{f(w)}{w-z}\,\mathrm{d}w \right| = \left| \int_0^1 f(z+\rho e^{2\pi i\theta})\,\mathrm{d}\theta \right| \leq \sup_{|z-w|=\rho} |f(w)| \leq f(z).$$

So we must have equality throughout. When we proved the supremum bound for the integral, we showed equality can happen only if the integrand is constant. So $|f(w)|$ is constant on the circle $|z-w| = \rho$, and is equal to $f(z)$. Since this is true for all $\rho \in (0,r)$, it follows that $|f|$ is constant on $B(z;r)$. Then the Cauchy-Riemann equations then entail that f must be constant, as you have shown in example sheet 1. □

2.3.1 Consequences of Cauchy's Theorem

Theorem (Liouville's theorem)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function (ie. holomorphic everywhere). If f is bounded, then f is constant.

Note that this is simply not true in $\mathbb{R}$. Like $\sin$.

Proof. Suppose $|f(z)| \leq M$ for all $z \in \mathbb{C}$. We fix $z_1, z_2 \in \mathbb{C}$, and estimate $|f(z_1) - f(z_2)|$ with the integral formula.

Let $R > \max\{2|z_1|, 2|z_2|\}$. By the integral formula, we know

$$\begin{aligned} |f(z_1) - f(z_2)| &= \left|\frac{1}{2\pi i}\int_{\partial B(0,R)} \left(\frac{f(w)}{w - z_1} - \frac{f(w)}{w - z_2}\right)\, \mathrm{d}w\right| \\ &= \left|\frac{1}{2\pi i}\int_{\partial B(0,R)} \frac{f(w)(z_1 - z_2)}{(w - z_1)(w - z_2)}\, \mathrm{d}w\right| \\ &\leq \frac{1}{2\pi} \cdot 2\pi R \cdot \frac{M|z_1 - z_2|}{(R/2)^2} \\ &= \frac{4M|z_1 - z_2|}{R}. \end{aligned}$$

□

Corollary (Fundamental theorem of algebra)**.** A non-constant complex polynomial has a root in $\mathbb{C}$.

Proof. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where $a_n \neq 0$ and $n > 0$. So P is non-constant. Thus, as $|z| \to \infty$, $|P(z)| \to \infty$. In particular, there is some R such that for $|z| > R$, we have $|P(z)| \geq 1$.

Now suppose for contradiction that P does not have a root in $\mathbb{C}$. Then consider

$$f(z) = \frac{1}{P(z)},$$

which is then an entire function. On $\overline{B(0, R)}$, we know f is certainly continuous, and hence bounded. Outside this ball, we get $|f(z)| \leq 1$. So $f(z)$ is constant, by Liouville's theorem. A contradiction. □

2.4 Taylor's Theorem

More goodies in Complex! In the complex field, taylor series *have* to be equal to the function:

Theorem (Taylor's theorem)**.** Let $f : B(a, r) \to \mathbb{C}$ be holomorphic. Then f has a convergent power series representation

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

on $B(a, r)$. Moreover, $c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_{\partial B(a,\rho)} \frac{f(z)}{(z-a)^{n+1}}\, \mathrm{d}z$ for any $0 < \rho < r$. Thus f is infinitely differentiable.

Proof. We'll use Cauchy's integral formula. If $|w-a|<\rho<r$, then

$$f(w)=\frac{1}{2\pi i}\int_{\partial B(a,\rho)}\frac{f(z)}{z-w}\,\mathrm{d}z.$$

Now (cf. the first proof of the Cauchy integral formula), we note that

$$\frac{1}{z-w}=\frac{1}{(z-a)\left(1-\frac{w-a}{z-a}\right)}=\sum_{n=0}^{n}\frac{(w-a)^n}{(z-a)^{n+1}}.$$

This series is uniformly convergent everywhere on the ρ disk, including its boundary. By uniform convergence, we can exchange integration and summation to get

$$f(w)=\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial B(a,\rho)}\frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z\right)(w-a)^n=\sum_{n=0}^{\infty}c_n(w-a)^n.$$

Since c_n does not depend on w, this is a genuine power series representation, and this is valid on any disk $B(a,\rho)\subseteq B(a,r)$.

Then the formula for c_n in terms of the derivative comes easily as that's the formula for the derivative of a power series. □

Note. Note that since every differentiable function is infinitely differentiable, if $f(z)$ satisfies that $\int\gamma f(z)=0$ for all γ on a domain, then we know it has a holomorphic antiderivative, thus itself is holomorphic too! This is called the *Morera's Theorem.*

2.5 Zeros

As everything in the course is now a power series, we proceed to find the zeros for them.

Definition (Order of zero)**.** Let $f:B(a,r)\to\mathbb{C}$ be holomorphic. Then we know we can write

$$f(z)=\sum_{n=0}^{\infty}c_n(z-a)^n$$

as a convergent power series. Then either all $c_n=0$, in which case $f=0$ on $B(a,r)$, or there is a least N such that $c_N\neq 0$ (N is just the smallest n such that $f^{(n)}(a)\neq 0$).

If $N>0$, then we say f has a *zero of order* N.

Lemma (Principle of isolated zeroes)**.** Let $f:B(a,r)\to\mathbb{C}$ be holomorphic and not identically zero. Then there exists some $0<\rho<a$ such that $f(z)\neq 0$ in the punctured neighbourhood $B(a,\rho)\setminus\{a\}$.

Proof. If $f(a)\neq 0$, then the result is obvious by continuity of f.

The other option is not too different. If f has a zero of order N at a, then we can write $f(z)=(z-a)^N g(z)$ with $g(a)\neq 0$. By continuity of g, g does not vanish on some small neighbourhood of a, say $B(a,\rho)$. Then $f(z)$ does not vanish on $B(a,\rho)\setminus\{a\}$. □

Using this, we can easily have the corollary below:

Corollary (Identity theorem). Let $U \subseteq \mathbb{C}$ be a domain, and $f, g : U \to \mathbb{C}$ be holomorphic. Let $S = \{z \in U : f(z) = g(z)\}$. If S contains a non-isolated point, ie. there exists some $w \in S$ such that for all $\varepsilon > 0$, $S \cap B(w, \varepsilon) \neq \{w\}$. Then $f = g$ on U.

Definition (Analytic continuation). Let $U_0 \subseteq U \subseteq \mathbb{C}$ be domains, and $f : U_0 \to \mathbb{C}$ be holomorphic. An *analytic continuation* of f is a holomorphic function $h : U \to \mathbb{C}$ such that $h|_{U_0} = f$, ie. $h(z) = f(z)$ for all $z \in U_0$. By the corollary above, we know if it exists, it is unique.

Note. An example would be extending $\sum z^n$ to $\mathbb{C}\ 1$ using $\frac{1}{1-z}$. There is really no good general way for doing problems like these, and you just have to try to write functions in different ways.

2.6 Singularities

Now we would consider the singularities around the holomorphic functions. And we would classify them using the propositions below:

Removable Singularity This class is really nice.

Proposition. Let U be a domain and $z_0 \in U$. If $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic, and f is bounded near z_0, then there exists an a such that $f(z) \to a$ as $z \to z_0$. Furthermore, if we define

$$g(z) = \begin{cases} f(z) & z \in U \setminus \{z_0\} \\ a & z = z_0 \end{cases},$$

then g is holomorphic on U.

Proof. Define a new function $h : U \to \mathbb{C}$ by

$$h(z) = \begin{cases} (z - z_0)^2 f(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases}.$$

Now h is holomorphic away from 0. Also, we know f is bounded near z_0. So suppose $|f(z)| < M$ in some neighbourhood of z_0. Then we have

$$\left| \frac{h(z) - h(z_0)}{z - z_0} \right| \leq |z - z_0| M.$$

So in fact h is also differentiable at z_0, and $h(z_0) = h'(z_0) = 0$. So near z_0, h has a Taylor series

$$h(z) = \sum_{n \geq 0} a_n (z - z_0)^n.$$

Since we are told that $a_0 = a_1 = 0$, we can define a $g(z)$ by

$$g(z) = \sum_{n \geq 0} a_{n+2} (z - z_0)^n,$$

defined on some ball $B(z_0, \rho)$, where the Taylor series for h is defined. By construction, on the punctured ball $B(z_0, \rho) \setminus \{z_0\}$, we get $g(z) = f(z)$. Moreover, $g(z) \to a_2$ as $z \to z_0$. So $f(z) \to a_2$ as $z \to z_0$.

Since g is a power series, it is holomorphic. So the result follows. □

Poles This class is also not very interesting, as we have:

Proposition. Let U be a domain, $z_0 \in U$ and $f : U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. Suppose $|f(z)| \to \infty$ as $z \to z_0$. Then there is a unique $k \in \mathbb{Z}_{\geq 1}$ and a unique holomorphic function $g : U \to \mathbb{C}$ such that $g(z_0) \neq 0$, and

$$f(z) = \frac{g(z)}{(z - z_0)^k}.$$

Proof. Let's pick some $\delta > 0$ such that $|f(z)| \geq 1$ for all $z \in B(z_0; \delta) \setminus \{z_0\}$. In particular, $f(z)$ is non-zero on $B(z_0; \delta) \setminus \{z_0\}$. So we can define

$$h(z) = \begin{cases} \frac{1}{f(z)} & z \in B(z_0; \delta) \setminus \{z_0\} \\ 0 & z = z_0 \end{cases}.$$

Since $|\frac{1}{f(z)}| \leq 1$ on $B(z_0; \delta) \setminus \{z_0\}$, by the removal of singularities, h is holomorphic on $B(z_0, \delta)$. Since h vanishes at the z_0, it has a unique definite order at z_0, ie. there is a unique integer $k \geq 1$ such that h has a zero of order k at z_0. In other words,

$$h(z) = (z - z_0)^k \ell(z),$$

for some holomorphic $\ell : B(z_0; \delta) \to \mathbb{C}$ and $\ell(z_0) \neq 0$.

Now by continuity of ℓ, there is some $0 < \varepsilon < \delta$ such that $\ell(z) \neq 0$ for all $z \in B(z_0, \varepsilon)$. Now define $g : B(z_0; \varepsilon) \to \mathbb{C}$ by

$$g(z) = \frac{1}{\ell(z)}.$$

Then g is holomorphic on this disc. By construction,we have

$$g(z) = \frac{1}{\ell(z)} = \frac{1}{h(z)} \cdot (z - z_0)^k = (z - z_0)^k f(z).$$

g was initially defined on $B(z_0; \varepsilon) \to \mathbb{C}$, but g admits an analytic continuation from $B(z_0; \varepsilon)$ to U. So done. □

We call k the degree of a pole. If it is 1, we call the pole *simple*.

Isolated Essential Singularity This is an *isolated* singularity that is neither removable nor a pole. $e^{\frac{1}{z}}$ has one at $z = 0$. This is bad. Very bad. The following theorems tell you how bad it is:

Theorem (Casorati-Weierstrass theorem)**.** Let U be a domain, $z_0 \in U$, and suppose $f : U \setminus \{z_0\} \to \mathbb{C}$ has an essential singularity at z_0. Then for all $w \in \mathbb{C}$, there is a sequence $z_n \to z_0$ such that $f(z_n) \to w$.

In other words, on any punctured neighbourhood $B(z_0; \varepsilon) \setminus \{z_0\}$, the image of f is dense in $\mathbb{C}$.

Proof. Suppose that on $U \setminus \{z_0\}$, $f(z)$ does not take values near $a \in \mathbb{C}$. Thus, if we take $g(z) = \frac{1}{f(z) - b}$, then g is bounded on $U \setminus \{z_0\}$ and holomorphic, so it can be extended to z_0. But then $f(z) = \frac{1}{g(z)} + b$ has a pole or a removable singularity. A contradiction. □

Actually, it is *very* bad.

Theorem (Picard's theorem)**.** If f has an isolated essential singularity at z_0, then there is some $b \in \mathbb{C}$ such that on each punctured neighbourhood $B(z_0; \varepsilon) \setminus \{z_0\}$, the image of f contains $\mathbb{C} \setminus \{b\}$.

The proof is not required.

Thus, we introduce some terminology for the holomorphic functions that we like:

Definition (Meromorphic function)**.** If U is a domain and $S \subseteq U$ is a finite or discrete set, a function $f : U \setminus S \to \mathbb{C}$ which is holomorphic and has (at worst) poles on S is said to be *meromorphic* on U.

2.7 Laurent Series

Now if we have a singularity at z_0, then we can't have a Taylor series. But we can have something close to it:

Theorem (Laurent series)**.** Let $0 \leq r < R < \infty$, and let

$$A = \{z \in \mathbb{C} : r < |z - a| < R\}$$

denote an annulus on $\mathbb{C}$.

Suppose $f : A \to \mathbb{C}$ is holomorphic. Then f has a (unique) convergent series expansion

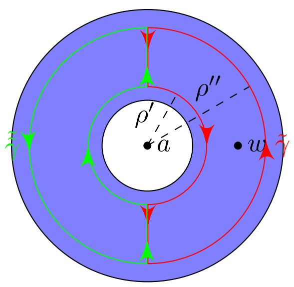
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n,$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial \overline{B(a, \rho)}} \frac{f(z)}{(z - a)^{n+1}} \,\mathrm{d}z$$

for $r < \rho < R$. Moreover, the series converges uniformly on compact subsets of the annulus.

Proof. Let $w \in A$. We let $r < \rho' < |w - a| < \rho'' < R$.



We let $\tilde{\gamma}$ be the contour containing w, and $\tilde{\tilde{\gamma}}$ be the other contour.

Now we apply the Cauchy integral formula to say

$$f(w) = \frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{f(z)}{z - w} \,\mathrm{d}z \qquad 0 = \frac{1}{2\pi i} \int_{\tilde{\tilde{\gamma}}} \frac{f(z)}{z - w} \,\mathrm{d}z.$$

So we get

$$f(w) = \frac{1}{2\pi i} \int_{\partial B(a, \rho'')} \frac{f(z)}{z - w}\;\mathrm{d}z - \frac{1}{2\pi i} \int_{\partial B(a, \rho')} \frac{f(z)}{z - w}\;\mathrm{d}z.$$

For the first integral, we have $w - a < z - a$. So

$$\frac{1}{z - w} = \frac{1}{z - a}\left(\frac{1}{1 - \frac{w - a}{z - a}}\right) = \sum_{n = 0}^\infty \frac{(w - a)^n}{(z - a)^{n + 1}},$$

which is uniformly convergent on $z \in \partial B(a, \rho'')$.

For the second integral, we have $w - a > z - a$. So

$$\frac{-1}{z - w} = \frac{1}{w - a}\left(\frac{1}{1 - \frac{z - a}{w - a}}\right) = \sum_{m = 1}^\infty \frac{(z - a)^{m - 1}}{(w - a)^m},$$

which is uniformly convergent for $z \in \partial B(a, \rho')$.

By uniform convergence, we can swap summation and integration. So we get

$$\begin{aligned}f(w) = \sum_{n = 0}^\infty &\left(\frac{1}{2\pi i} \int_{\partial B(a, \rho'')} \frac{f(z)}{(z - a)^{n + 1}}\;\mathrm{d}z\right) (w - a)^n\\ &+ \sum_{m = 1}^\infty \left(\frac{1}{2\pi i} \int_{\partial B(a, \rho')} \frac{f(z)}{(z - a)^{-m + 1}}\;\mathrm{d}z\right) (w - a)^{-m}.\end{aligned}$$

Now we substitute $n = -m$ in the second sum, and get

$$f(w) = \sum_{n = -\infty}^\infty \tilde{c}_n (w - a)^n,$$

for the integrals $\tilde{c}_n$. However, some of the coefficients are integrals around the ρ'' circle, while the others are around the ρ' circle. This is not a problem. For any $r < \rho < R$, these circles are convex deformations (which mean they can be deformed to another curve continuously) of $|z - a| = \rho$ inside the annulus A. So

$$\int_{\partial B(a, \rho)} \frac{f(z)}{(z - a)^{n + 1}}\;\mathrm{d}z$$

is independent of ρ as long as $\rho \in (r, R)$. □

Now we have this nice series formula, but how do we actually calculate these coefficients. Well we can go for the formula, but this is brutal. Is there any better way? Unfortunately nope. Here is an example of the guesswork:

Example. We know

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

defines a holomorphic function, with a radius of convergence of ∞. Now consider

$$\operatorname{cosec} z = \frac{1}{\sin z},$$

which is holomorphic except for $z = k\pi$, with $k \in \mathbb{Z}$. So $\operatorname{cosec} z$ has a Laurent series near $z = 0$. Using

$$\sin z = z\left(1 - \frac{z^2}{6} + O(z^4)\right),$$

we get

$$\operatorname{cosec} z = \frac{1}{z}\left(1 + \frac{z^2}{6} + O(z^4)\right).$$

From this, we can read off that the Laurent series has $c_n = 0$ for all $n \leq -2$, $c_{-1} = 1$, $c_1 = \frac{1}{5}$. By periodicity, cosec has a simple pole at all other singularities.

3 Residue Calculus

This section focuses on how to apply the theory in the previous sections to real work. To do this, we need some helpful definitions first:

Definition (Residue)**.** Let $f : B(a,r)\setminus\{a\} \to \mathbb{C}$ be holomorphic, with Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n.$$

Then the *residue* of f at a is

$$\operatorname{Res}(f,a) = \operatorname{Res}_f(a) = c_{-1} = \frac{1}{2\pi i}\int_{\overline{\partial B(a,\rho)}} f(z)\,\mathrm{d}z$$

Now before we use this exciting fact in integrating, we need to prove some lemmas:

Lemma. Let $\gamma : [a,b] \to \mathbb{C}$ be a continuous closed curve, and pick a point $w \in \mathbb{C}\setminus\operatorname{Im}(\gamma)$. Then there are continuous functions $r : [a,b] \to \mathbb{R} > 0$ and $\theta : [a,b] \to \mathbb{R}$ such that

$$\gamma(t) = w + r(t)e^{i\theta(t)}.$$

Proof. Clearly $r(t) = |\gamma(t) - w|$ exists and is continuous, since it is the composition of continuous functions. Note that this is never zero since $\gamma(t)$ is never w. The actual content is in defining θ.

To define $\theta(t)$, we for simplicity assume $w = 0$. Furthermore, by considering instead the function $\frac{\gamma(t)}{r(t)}$, which is continuous and well-defined since r is never zero, we can assume $|\gamma(t)| = 1$ for all t.

Since $\gamma : [a,b] \to \mathbb{C}$ is continuous, it is uniformly continuous, and we can find a subdivision

$$a = a_0 < a_1 < \cdots < a_m = b,$$

such that if $s, t \in [a_{i-1}, a_i]$, then $|\gamma(s) - \gamma(t)| < \sqrt{2}$, and hence we can define $\gamma(s)$ and $\gamma(t)$ continuously (remember we can define $\log$ continuously on any half plane, and this defines a half plane).

So we define $\theta_j : [a_{j-1}, a_j] \to \mathbb{R}$ such that

$$\gamma(t) = e^{i\theta_j(t)}$$

for $t \in [a_{j-1}, a_j]$, and $1 \leq j \leq n-1$.

On each region $[a_{j-1}, a_j]$, this gives a continuous argument function. We cannot immediately extend this to the whole of $[a, b]$, since it is entirely possible that $\theta_j(a_j) \neq \theta_{j+1}(a_j)$, so we have discontinuity. However, we do know that $\theta_j(a_j)$ and $\theta_{j+1}(a_j)$ are both values of the argument of $\gamma(a_j)$. So they must differ by an integer multiple of 2π, say $2n\pi$. Then we can just replace θ_{j+1} by $\theta_{j+1} - 2n\pi$, which is an equally valid argument function, and then the two functions will agree at a_j.

Hence, for $j > 1$, we can successively re-define θ_j such that the resulting map θ is continuous. Then we are done. □

Now we define winding number, which, unsurprisingly, is the number of times you go around a certain point:

Definition (Winding number). Given a continuous path $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(a) = \gamma(b)$ and $w \notin \operatorname{Im}(\gamma)$, the *winding number* of γ about w is

$$\frac{\theta(b) - \theta(a)}{2\pi},$$

where $\theta : [a, b] \to \mathbb{R}$ is a continuous function as above. This is denoted by $I(\gamma, w)$.

Now here is our first useful result:

Lemma. Suppose $\gamma : [a, b] \to \mathbb{C}$ is a piecewise C^1-smooth closed path, and $w \notin \operatorname{Im}(\gamma)$. Then

$$I(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - w} \, \mathrm{d}z.$$

Proof. Let $\gamma(t) - w = r(t)e^{i\theta(t)}$, with now r and θ piecewise C^1-smooth. Then

$$\int_\gamma \frac{1}{z - w} \, \mathrm{d}z = \int_a^b \left(\frac{r'(t)}{r(t)} + i\theta'(t) \right) \, \mathrm{d}t = [\ln r(t) + i\theta(t)]_a^b = 2\pi i I(\gamma, w).$$

□

Remember the idea that we can continuously deform curve and not change the integral? Well we have a formal definition now:

Definition (Homotopy of closed curves). Let $U \subseteq \mathbb{C}$ be a domain, and let $\phi : [a, b] \to U$ and $\psi : [a, b] \to U$ be piecewise C^1-smooth closed paths. A *homotopy* from $\phi : \psi$ is a continuous map $F : [0, 1] \times [a, b] \to U$ such that

$$F(0, t) = \phi(t), \quad F(1, t) = \psi(t),$$

and moreover, for all $s \in [0, t]$, the map $t \mapsto F(s, t)$ viewed as a map $[a, b] \to U$ is closed and piecewise C^1-smooth.

Now since we can now continuously deform curves, we can upgrade Cauchy Theorem to its usual version:

Corollary (Cauchy's theorem V1.0). Let U be a simply connected domain, and let $f : U \to \mathbb{C}$ be holomorphic. If γ is any piecewise C^1-smooth closed curve in U, then

$$\int_\gamma f(z) \, \mathrm{d}z = 0.$$

3.1 Residue Theorem. Or Doing the Integrals the Right™ way

Theorem (Cauchy's residue theorem). Let U be a simply connected domain, and $\{z_1, \cdots, z_k\} \subseteq U$. Let $f : U \setminus \{z_1, \cdots, z_k\} \to \mathbb{C}$ be holomorphic. Let $\gamma : [a, b] \to U$ be a piecewise C^1-smooth closed curve such that $z_i \neq \operatorname{Im}(\gamma)$ for all i. Then

$$\frac{1}{2\pi i} \int_\gamma f(z)\, \mathrm{d}z = \sum_{j=1}^{k} I(\gamma, z_i) \operatorname{Res}(f; z_i).$$

The Cauchy integral formula and simply-connected Cauchy are special cases of this.

Proof. At each z_i, f has a Laurent expansion

$$f(z) = \sum_{n \in \mathbb{Z}} c_n^{(i)} (z - z_i)^n,$$

valid in some neighbourhood of z_i. Let $g_i(z)$ be the principal part, namely

$$g_i(z) = \sum_{n=-\infty}^{-1} c_n^{(i)} (z - z_i)^n.$$

From the proof of the Laurent series, we know $g_i(z)$ gives a holomorphic function on $U \setminus \{z_i\}$.

We now consider $f - g_1 - g_2 - \cdots - g_k$, which is holomorphic on $U \setminus \{z_1, \cdots, z_k\}$, and has a *removable* singularity at each z_i. So

$$\int_\gamma (f - g_1 - \cdots - g_k)(z)\, \mathrm{d}z = 0,$$

by simply-connected Cauchy. Hence we know

$$\int_\gamma f(z)\, \mathrm{d}z = \sum_{j=1}^{k} \int_\gamma g_j(z)\, \mathrm{d}z.$$

For each j, we use uniform convergence of the series $\sum_{n \leq -1} c_n^{(j)} (z - z_j)^n$ on compact subsets of $U \setminus \{z_j\}$, and hence on γ, to write

$$\int_\gamma g_j(z)\, \mathrm{d}z = \sum_{n \leq -1} c_n^{(j)} \int_\gamma (z - z_j)^n\, \mathrm{d}z.$$

However, for $n \neq -1$, the function $(z - z_j)^n$ has an antiderivative, and hence the integral around γ vanishes. So this is equal to

$$c_{-1}^{(j)} \int_\gamma \frac{1}{z - z_j}\, \mathrm{d}z.$$

But $c_{-1}^{(j)}$ is by definition the residue of f at z_j, and the integral is just the integral definition of the winding number (up to a factor of $2\pi i$). So we get

$$\int_\gamma f(z)\, \mathrm{d}z = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f; z_j) I(\gamma, z_j).$$

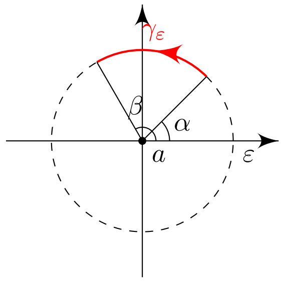
So done. □

3.1.1 Useful Results for Integration

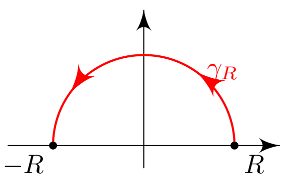
This is a *non-exhaustive* list for Integration tips using the Residue Theorem that should be very useful on the test:

- Useful Lemmas
  - Let f have a pole at a. Then we have the following results:
    - If the pole is simple, then $\mathrm{Res}(f, a) = \lim_{z\to a}(z-a)f(z)$.
    - If near a, we can write $f(z) = \frac{g(z)}{h(z)}$, where $g(a) \neq 0$ and h has a simple zero at a, and g, h are holomorphic on $B(a, \varepsilon) \setminus \{a\}$, then
    $$\mathrm{Res}(f, a) = \frac{g(a)}{h'(a)}.$$
    - If $f(z) = \frac{g(z)}{(z-a)^k}$ near a, with $g(a) \neq 0$ and g is holomorphic, then
    $$\mathrm{Res}(f, a) = \frac{g^{(k-1)}(a)}{(k-1)!}.$$
  - Residual theorem kinds of works in sectors:

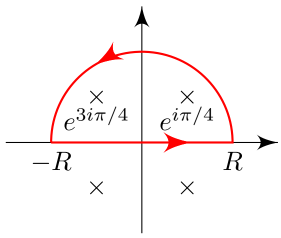
    Lemma. Let $f : B(a, r) \setminus \{a\} \to \mathbb{C}$ be holomorphic, and suppose f has a simple pole at a. We let $\gamma_\varepsilon : [\alpha, \beta] \to \mathbb{C}$ be given by
    $$t \mapsto a + \varepsilon e^{it}.$$



    Then
    $$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z = (\beta - \alpha)\cdot i \cdot \mathrm{Res}(f, a).$$
  - **Lemma** (Jordan's lemma)**.** Let f be holomorphic on a neighbourhood of infinity in $\mathbb{C}$, ie. on $\{|z| > r\}$ for some $r > 0$. Assume that $zf(z)$ is bounded in this region. Then for $\alpha > 0$, we have
    $$\int_{\gamma_R} f(z)e^{i\alpha z}\,\mathrm{d}z \to 0$$
    as $R \to \infty$, where $\gamma_R(t) = Re^{it}$ for $t \in [0, \pi]$ is the *not closed* semicircle.



Half Circle Contour (HCF) For quasi-normal integrals like $\int_0^\infty \frac{1}{1+x^4}\,\mathrm{d}x$, we treat them as complex integrals and integrate them over a half-circular arc like this one:

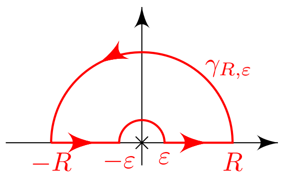


Then the integral along the whole curve is obtained by the residual theorem. The circular arc part? We let R go to infinity and it vanishes.

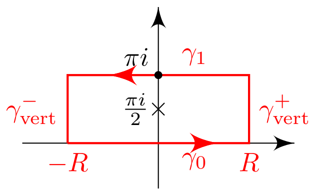
Trig Functions Help! Trig functions don't just vanish when R gets large in the Half Circle Contour. What should we do?

- For functions like $f(x) = \frac{\cos x}{1+x+x^2}$, just change cos/sin to e^{ix} and *then* treat them with the HCF. And then take the real/imaginary part.
- For functions more like $\frac{1}{1+\sin^2(x)}$, Take $\sin x = \frac{e^x - e^{-x}}{2i}$ and $\cos x = \frac{e^x + e^{-x}}{2}$. Then treat it with the appropriate contour.

Contour with Singularity What if $f(z) = \frac{e^z}{z}$? We take the following contour:

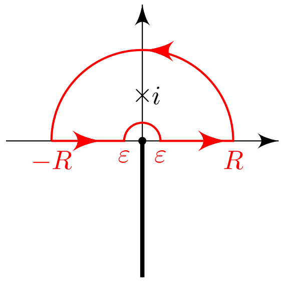


A LOT of Poles Now $f(z) = \frac{e^{az}}{\cosh z}$ has poles at all $z = (n + \frac{1}{2})i\pi$. To calculate its integral from $-\infty$ to ∞, we use:



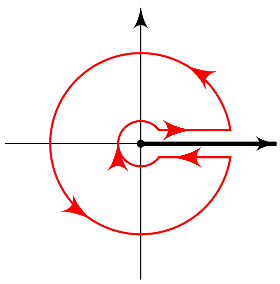
Logarithms

- Remember to take the branch cut! For example, if we want to comput $\frac{\log x}{1+x^2}$, we can take the branch cut along the negative imaginary axis to avoid cutting through contours:



- Separate the logarithm! When you are doing logarithms like $\log(x^2+1)$, break it into $\log(x+i)$ and $\log(x-i)$ (pay attention to the branch taken) to make your life easier.

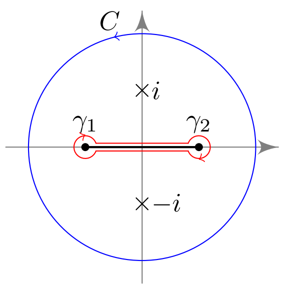
Keyhole Contour If we would like to compute the integral for $\frac{\sqrt{x}}{x^2+ax+b}$ then we take the keyhole contour:



The integrals on the circle vanish, but viewing $\sqrt{z} = e^{\frac{1}{2}\log z}$, on the two pieces of the contour along $\mathbb{R}_{\geq 0}$, $\log z$ differs by $2\pi i$. So $\sqrt{z}$ changes sign. This cancels with the sign change arising from going in the wrong direction. Therefore the residue theorem says

$$2\pi i \sum \text{residues inside contour} = 2\int_0^\infty \frac{\sqrt{x}}{x^2+ax+b}\,\mathrm{d}x.$$

Dogbone Contour If we are integrating functions like $\frac{\sqrt{1-x^2}}{1+x^2}$, we have two problems at 1 and -1. We use the dogbone contour:



Where γ_1, γ_2, and C go the zero, and the whole blue plus red curve add together to the residues at i and $-i$.

Sum of Infinite Series The tactic is to surround a lot of poles. We illustrate this using an example:

Example. We want to prove that

$$\sum_{n\geq 1}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

We consider the function $f(z)=\frac{\pi\cot(\pi z)}{z^2}$, which is holomorphic on $\mathbb{C}$ except for simple poles at $\mathbb{Z}\setminus\{0\}$, and a triple pole at 0.

We can check that at $n\in\mathbb{Z}\setminus\{0\}$, we can write

$$f(z)=\frac{\pi\cos(\pi z)}{z^2}\cdot\frac{1}{\sin(\pi z)},$$

where the second term has a simple zero at n, and the first is non-vanishing at $n\neq 0$. Then we have compute

$$\operatorname{Res}(f;n)=\frac{\pi\cos(\pi n)}{n^2}\cdot\frac{1}{\pi\cos(\pi n)}=\frac{1}{n^2}.$$

Note that the reason why we have those funny π's all around the place is so that we can get this nice expression for the residue.

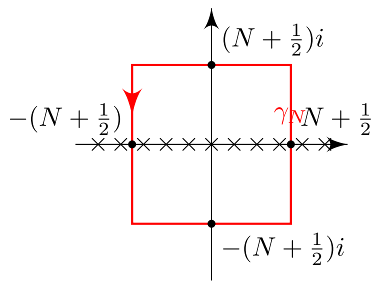
At $z=0$, we get

$$\cot(z)=\left(1-\frac{z^2}{2}+O(z^4)\right)\left(z-\frac{z^3}{3}+O(z^5)\right)^{-1}=\frac{1}{z}-\frac{z}{3}+O(z^2).$$

So we get

$$\frac{\pi\cot(\pi z)}{z^2}=\frac{1}{z^3}-\frac{\pi^2}{3z}+\cdots$$

So the residue is $-\frac{\pi^2}{3}$. Now we consider the following square contour:



Since we don't want the contour itself to pass through singularities, we make the square pass through $\pm\left(N+\frac{1}{2}\right)$. Then the residue theorem says

$$\int_{\gamma_N} f(z)\,\mathrm{d}z=2\pi i\left(2\sum_{n=1}^{N}\frac{1}{n^2}-\frac{\pi^2}{3}\right).$$

We can thus get the desired series if we can show that

$$\int_{\gamma_N} f(z)\,\mathrm{d}z\to 0 \text{ as } n\to\infty.$$

We first note that

$$\begin{aligned}\left|\int_{\gamma_N} f(z)\,\mathrm{d}z\right| &\le \sup_{\gamma_N}\left|\frac{\pi\cot\pi z}{z^2}\right| 4(2N+1)\\ &\le \sup_{\gamma_N}|\cot\pi z|\frac{4(2N+1)\pi}{\left(N+\frac{1}{2}\right)^2}\\ &= \sup_{\gamma_N}|\cot\pi z| O(N^{-1}).\end{aligned}$$

So everything is good if we can show $\sup_{\gamma_N}|\cot\pi z|$ is bounded as $N\to\infty$.

On the vertical sides, we have

$$z=\pi\left(N+\frac{1}{2}\right)+iy,$$

and thus

$$|\cot(\pi z)|=|\tan(i\pi y)|=|\tanh(\pi y)|\le 1,$$

while on the horizontal sides, we have

$$z=x\pm i\left(N+\frac{1}{2}\right),$$

and

$$|\cot(\pi z)|\le\frac{e^{\pi(N+1/2)}+e^{-\pi(N+1/2)}}{e^{\pi(N+1/2)}-e^{-\pi(N+1/2)}}=\coth\left(N+\frac{1}{2}\right)\pi.$$

While it is not clear at first sight that this is bounded, we notice $x\mapsto\coth x$ is decreasing and positive for $x\ge 0$. So we win.

3.2 Rouche's Theorem

Here we move away from computing (fun!) integrals, and look at another application of Residue's theorem:

Theorem (Argument principle)**.** Let U be a simply connected domain, and let f be meromorphic on U. Suppose in fact f has finitely many zeroes $z_1,\cdots,z_k$ and finitely many poles $w_1,\cdots,w_\ell$. Let γ be a piecewise-C^1 closed curve such that $z_i,w_j\notin\mathrm{Im}(\gamma)$ for all i,j. Then

$$\frac{1}{2\pi i}\int_\gamma\frac{f'(z)}{f(z)}\,\mathrm{d}z=\sum_{i=1}^{k}\mathrm{ord}(f;z_i)I_\gamma(z_i)-\sum_{j=1}^{\ell}\mathrm{ord}(f,w_j)I(\gamma,w_j).$$

Proof. By the residue theorem, we have

$$\frac{1}{2\pi i}\int_\gamma\frac{f'(z)}{f(z)}\,\mathrm{d}z=\sum_{z\in U}\mathrm{Res}\left(\frac{f'}{f},z\right)I(\gamma,z),$$

where we sum over all zeroes and poles of z. Note that outside these zeroes and poles, the function $\frac{f'(z)}{f(z)}$ is holomorphic.

Now at each z_i, if $f(z) = (z - z_j)^k g(z)$, with $g(z_j) \neq 0$, then by direct computation, we get

$$\frac{f'(z)}{f(z)} = \frac{k}{z - z_j} + \frac{g'(z)}{g(z)}.$$

Since at z_j, g is holomorphic and non-zero, we know $\frac{g'(z)}{g(z)}$ is holomorphic near z_j. So

$$\operatorname{Res}\left(\frac{f'}{f}, z_j\right) = k = \operatorname{ord}(f, z_j).$$

Analogously, by the same proof, at the w_i, we get

$$\operatorname{Res}\left(\frac{f'}{f}, w_j\right) = -\operatorname{ord}(f; w_j).$$

So done. □

Note. The reason this is called the argument principle is because:

Let $\Gamma = f \circ \gamma$, and since γ does not contain zeroes or poles of f, Γ is a piecewise-C^1 closed curve in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Making a simple substitution $w = f(z)$, we find

$$2\pi i \int_\gamma \frac{f'(z)}{f(z)}\, \mathrm{d}z = \frac{1}{2\pi i} \int_\Gamma \frac{\mathrm{d}w}{w}$$

This is, by definition, the winding number $I(\Gamma; 0)$. So the argument principle says the difference between the total order of the zeroes and poles encircled by the curve is given by the winding number (argument) of the composite curve $\Gamma = f \circ \gamma$ around the origin.

What this leads to is an interesting corollary:

Corollary (Rouchés theorem)**.** Let U be a domain and γ a closed curve which bounds a domain in U (the key case is when U is simply connected and γ is a simple closed curve). Let f, g be holomorphic on U, and suppose $|f| > |g|$ for all $z \in \operatorname{Im}(\gamma)$. Then f and $f + g$ have the same number of zeroes in the domain bound by γ, when counted with multiplicity.

Proof. If $|f| > |g|$ on γ, then f and $f + g$ cannot have zeroes on the curve γ. We let

$$h(z) = \frac{f(z) + g(z)}{f(z)} = 1 + \frac{g(z)}{f(z)}.$$

This is a natural thing to consider, since zeroes of $f + g$ is zeroes of h, while poles of h are zeroes of f. Note that by assumption, for all $z \in \gamma$, we have

$$h(z) \in B(1, 1) \subseteq \{z : \Re z > 0\}.$$

Therefore $h \circ \gamma$ is a closed curve in the half-plane $\{z : \Re z > 0\}$. So $I(h \circ \gamma; 0) = 0$. Then by the argument principle, h must have the same number of zeros as poles in D, when counted with multiplicity (note that the winding numbers are all $+1$).

Thus, as the zeroes of h are the zeroes of $f + g$, and the poles of h are the poles of f, the result follows. □

Example. Consider the function $z^6 + 6z + 3$. This has three roots (with multiplicity) in $\{1 < |z| < 2\}$. To show this, note that on $|z| = 2$, we have

$$|z|^4 = 16 > 6|z| + 3 \geq |6z + 3|.$$

So if we let $f(z) = z^4$ and $g(z) = 6z + 3$, then f and $f + g$ have the same number of roots in $\{|z| < 2\}$. Hence all four roots lie inside $\{|z| < 2\}$.

On the other hand, on $|z| = 1$, we have

$$|6z| = 6 > |z^4 + 3|.$$

So $6z$ and $z^6 + 6z + 3$ have the same number of roots in $\{|z| < 1\}$. So there is exactly one root in there, and the remaining three must lie in $\{1 < |z| < 2\}$ (the bounds above show that $|z|$ cannot be exactly 1 or 2). So done.

Definition (Local degree)**.** Let $f : B(a, r) \to \mathbb{C}$ be holomorphic and non-constant. Then the *local degree* of f at a, written $\deg(f, a)$ is the order of the zero of $f(z) - f(a)$ at a.

If we take the Taylor expansion of f about a, then the local degree is the degree of the first non-zero term after the constant term.

Now if we want to know if the roots are distinct or not, we need the local degree theorem:

Lemma. The local degree is given by

$$\deg(f, a) = I(f \circ \gamma, f(a)),$$

where

$$\gamma(t) = a + re^{it},$$

with $0 \leq t \leq 2\pi$, for $r > 0$ sufficiently small.

Proof. Note that by the identity theorem, we know that, $f(z) - f(a)$ has an isolated zero at a (since f is non-constant). So for sufficiently small r, the function $f(z) - f(a)$ does not vanish on $\overline{B(a, r)} \setminus \{a\}$. If we use this r, then $f \circ \gamma$ never hits $f(a)$, and the winding number is well-defined. The result then follows directly from the argument principle. □

Proposition (Local degree theorem)**.** Let $f : B(a, r) \to \mathbb{C}$ be holomorphic and non-constant. Then for $r > 0$ sufficiently small, there is $\varepsilon > 0$ such that $f(z) = w$ has exactly $\deg(f, a)$ distinct solutions for $w \in B(f(a), \varepsilon) \setminus \{f(a)\}$.

Proof. We pick $r > 0$ such that $f(z) - f(a)$ and $f'(z)$ don't vanish on $B(a, r) \setminus \{a\}$. We let $\gamma(t) = a + re^{it}$. Then $f(a) \notin \operatorname{Im}(f \circ \gamma)$. So there is some $\varepsilon > 0$ such that

$$B(f(a), \varepsilon) \cap \operatorname{Im}(f \circ \gamma) = \emptyset.$$

We now let $w \in B(f(a), \varepsilon)$. Then the number of zeros of $f(z) - w$ in $B(a, r)$ is just $I(f \circ \gamma, w)$, by the argument principle. This is just equal to $I(f \circ \gamma, f(a)) = \deg(f, a)$, by the invariance of $I(\Gamma, *)$ as we move $*$ in a component $\mathbb{C} \setminus \Gamma$.

Now if $w \neq f(a)$, since $f'(z) \neq 0$ on $B(a, r) \setminus \{a\}$, all roots of $f(z) - w$ must be simple. So there are exactly $\deg(f; a)$ distinct zeros. □

The local degree theorem says the equation $f(z) = w$ has $\deg(f, a)$ roots for w sufficiently close to $f(a)$. In particular, we know there *are* some roots. So $B(f(a), \varepsilon)$ is contained in the image of f. So we get the following result:

Corollary (Open mapping theorem)**.** Let U be a domain and $f : U \to \mathbb{C}$ is holomorphic and non-constant, then f is an open map, ie. for all open $V \subseteq U$, we get that $f(V)$ is open.

Proof. This is an immediate consequence of the local degree theorem. It suffices to prove that for every $z \in U$ and $r > 0$ sufficiently small, we can find $\varepsilon > 0$ such that $B(f(a), \varepsilon) \subseteq f(B(a, r))$. This is true by the local degree theorem. □

Recall that Liouville's theorem says every holomorphic $f : \mathbb{C} \to B(0, 1)$ is constant. However, for any other simply connected domain, we know there are some interesting functions we can write down.

Corollary. Let $U \subseteq \mathbb{C}$ be a simply connected domain, and $U \neq \mathbb{C}$. Then there is a non-constant holomorphic function $U \to B(0, 1)$.

This is a weak form of the Riemann mapping theorem, which says that there is a *conformal equivalence* to $B(0, 1)$.

Proof. We let $q \in \mathbb{C} \setminus U$, and let $\phi(z) = z - q$. So $\phi : U \to \mathbb{C}$ is non-vanishing. It is also clearly holomorphic and non-constant. By an exercise (possibly on the example sheet), there is a holomorphic function $g : U \to \mathbb{C}$ such that $\phi(z) = e^{g(z)}$ for all z. In particular, our function $\phi(z) = z - q : U \to \mathbb{C}^*$ can be written as $\phi(z) = h(z)^2$, for some function $h : U \to \mathbb{C}^*$ (by letting $h(z) = e^{\frac{1}{2}g(z)}$).

We let $y \in h(U)$, and then the open mapping theorem says there is some $r > 0$ with $B(y, r) \subseteq h(U)$. But notice ϕ is injective by observation, and that $h(z_1) = \pm h(z_2)$ implies $\phi(z_1) = \phi(z_2)$. So we deduce that $B(-y, r) \cap h(U) = \emptyset$ (note that since $y \neq 0$, we have $B(y, r) \cap B(-y, r) = \emptyset$ for sufficiently small r).

Now define

$$f : z \mapsto \frac{r}{2(h(z) + y)}.$$

This is a holomorphic function $f : U \to B(0, 1)$, and is non-constant. □