Asymptotic Methods

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For Later

Contents

1 Definitions and Motivations

We introduce some definitions.

Definition. We say that $f(x)$ is *asymptotic* to $g(x)$ at $x = x_0$, denoted $f(x) \sim g(x)$, iff:

$$f(x) - g(x) = o(g(x))$$

for $x \to x_0$.

We also say that $f(x)$ is *asymptotic* to $\sum a_n x^n$, a power series, at $x = 0$, iff:

$$|f(x) - \sum^{N} a_n x^n| = o(x^N)$$

for all $N \in \mathbb{N}$.

Note. Note these two definitions are *NOT* compatible. You can have a nonzero function asymptotic to the 0 *series*, but not the function. Take $f(x) = e^{-\frac{1}{x^2}}$ as an example.

Now we introduce an important function.

Definition (Gamma Function)**.** The gamma function is defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t$$

And it is defined for $\Re(z) > 0$.

This has an important property for $\Re(z) > 1$:

$$\begin{aligned}
\Gamma(z) &= \int_0^\infty -t^{z-1} \mathrm{d}(e^{-t}) \\
&= [-t^{z-1}e^{-t}]_0^\infty + \int_0^\infty e^{-t}(z-1)t^{z-2}\mathrm{d}t \\
&= (z-1)\Gamma(z-1)
\end{aligned}$$

other cool good to now features are:

(i) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. and thus $\Gamma(n + \frac{1}{2}) = (n - \frac{1}{2}) \cdots (\frac{1}{2})\sqrt{pi}$.

2 Asymptotic Relations

Now we try to solve a simple differential equation:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = xy$$

Using the power series method $y = \sum a_n x^n$. We can easily generate the recurrence relation:

$$(n+3)(n+2)a_{n+3} = a_n$$

with $a_2 = 0$. Now we separate two cases:

$a_1 = 0$ Now we have:

$$a_{3n} = \frac{a_0}{3^n n! e^n (n - \frac{1}{3}) \cdots (\frac{2}{3})} = \frac{a_0 \Gamma(\frac{2}{3})}{3^{2n} n! \Gamma(n + \frac{2}{3})}$$

$a_0 = 0$ Similarly:

$$a_{3n+1} = \frac{a_1}{e^{2n} n! \Gamma(n + \frac{4}{3})}$$

Thus the conclusion is that:

$$y(x) = \alpha y_I(x) + \beta y_{II}(x)$$

where $y_I(x) = \sum \frac{x^{3n}}{3^{2n} n! \Gamma(n + \frac{2}{3})}$ and $y_{II}(x) = \sum \frac{x^{3n+1}}{3^{2n} n! \Gamma(n + \frac{4}{3})}$.

Now we do this again(Wait....why are we doing this? This isn't IA).Try:

$$x^2 y'' + (1 + 3x)y' + y = 0$$

We would reach that the power series solution is that $y = a_0 \sum (-1)^n n! x^n$, which has 0 radius of convergence! How do we fix this? Magic!

$$\begin{aligned}\sum (-1)^n n! x^n &= \sum (-1)^n \int_0^\infty (xt)^n e^{-t} \mathrm{d}t \\ &= \int_0^\infty e^{-t} \sum (-1)^n (xt)^n \mathrm{d}t \\ &= \int_0^\infty \frac{e^{-t}}{1 + xt} \mathrm{d}t\end{aligned}$$

Now if you have *any* recollection fo analysis, a hundred alarms should be ringing in your head right now. This isn't correct. You can't jut switch sums and integrals like that. That geometric series doesn't converge. Please, ignore these problems for now.

2.1 Stokes Lines

Let us consider $\sinh z^{-1}$ as $z \to 0$. Let $z = re^{i\theta}$. Then $z^{-1} = \frac{1}{r}(\cos\theta - i \sin\theta)$. So we have:

$$\sinh z^{-1} = \frac{1}{2}\left(e^{\frac{1}{r}(\cos\theta - i\sin\theta)} - e^{\frac{-1}{r}(\cos\theta - i\sin\theta)}\right)$$

If $\cos\theta > 0$, then the first term is *dominant* the second term is *recessive*, and vice versa. Then we see that the boundary in which one changes from dominant to recessive is $\cos\theta = 0$, which is the y axis in the complex plane. This line is called a *Stokes line*.

Thus we have

$$\sinh z^{-1} \sim \frac{1}{2} e^{z^{-1}} \qquad \sinh z^{-1} \sim \frac{1}{2} e^{-z^{-1}}$$

The first relation is true when $\arg z \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and the second when $\arg z \in (\frac{\pi}{2}, \frac{3\pi}{2})$

On Stokes line the two terms are comparable, so strictly speaking the relationships are true if $z \to 0$ in any way such that they are *positively* bounded away from the Stokes line-meaning they can't approach tangent to it.

Similarly we look at $\tanh z$ when $z \to \infty$. By the same argument, when $\cos\theta > 0$, then e^z term is dominant and $|e^{-z}| < 1$, so:

$$\tanh z = (1 - e^{-2z})(1 + e^{-2z})^{-1} = (1 - e^{-2z})(1 - e^{-2z} + e^{-4z} + \cdots)$$

Thus we have, in this sector:

$$\tanh z \sim 1 - 2e^{-2z} + 2e^{-4z} - \cdots$$

Note. For these asymptotic expansions, we can add, multiply and divide (non-zero) them, and we can integrate them but *NOT* differentiate them!

3 Asymptotic Expansions of Integrals

Now remember when we handwaved? Yes, the following integral:

$$\int_0^\infty \frac{e^{-t}}{1+xt}\mathrm{d}t = \lambda \int_0^\infty \frac{e^{-\lambda u}}{1+u}\mathrm{d}u$$

where $\lambda = \frac{1}{x} \to \infty$. The exponential $e^{-\lambda u}$ has the effect of localizing the integral at place where $e^{-\lambda u}$ is maximized. This is called the *Laplace principle*. So the integral is *asymptotically* localized at $u = 0$ [We can check that $\int_\epsilon^R \frac{e^{-\lambda u}}{1+u} \sim o(x^N)$ for all N, and all $\epsilon > 0$] Thus, $\int_0^\infty \frac{e^{-\lambda u}}{1+u}\mathrm{d}u$ has the same asymptotic expansion as $\int_0^\epsilon \frac{e^{-\lambda u}}{1+u}\mathrm{d}u$ for $|epsilon$ small. Then we can expand $\frac{1}{1+u}$ without handwaving to get:

$$\int_0^\epsilon \frac{e^{-\lambda u}}{1+u}\mathrm{d}u \sim \int_0^\epsilon e^{-\lambda u}(\sum(-1)^n u^n)$$

But now again by Laplace principle, $\int_0^\epsilon e^{-\lambda u}u^n\mathrm{d}u \sim \int_0^\infty e^{-\lambda u}u^n\mathrm{d}u$, so:

$$\lambda\int_0^\infty \frac{e^{-\lambda u}}{1+u}\mathrm{d}u \sim \sum\int_0^\infty \lambda e^{-\lambda u}(-1)^n u^n\mathrm{d}u$$

Using the definition of $\Gamma(z) = (-1)^n\lambda^n\int_0^\infty \lambda e^{-\lambda u}u^n\mathrm{d}u$, we have:

$$\lambda\int_0^\infty \frac{e^{-\lambda u}}{1+u}\mathrm{d}u \sim \sum(-1)^n x^n n!$$

Now we formally define the laplace principle:

Definition. The laplace integrals are:

$$\int_a^b f(t)e^{x\phi(t)}\mathrm{d}t$$

where $x \to \infty$.

Proposition (Laplace localization principle)**.** Assume there exists $t_0 \in (a, b)$ where $\phi(t_0) = \max_{[a,b]}\phi$, and t_0 is an *isolated* maximum, then we can ignore the contributions in $[a, t_0\epsilon]$ and $t_0 + \epsilon, b]$ since:

$$|\int_a^{t_0-\epsilon} f(t)e^{x\phi(t)}\mathrm{d}t| \le e^{x(M-a)}\int_a^b |f|\mathrm{d}t$$

Since t_0 is isolated. Thus the contribution here is $e^{-ax}O(e^{xM})$, but we are going $x \to \infty$, so we can ignore it.

Example (Direct Expansion).

$$I = \int_{-\infty}^{\infty} e^{-x\cosh u} \mathrm{d}u$$

We know that the asymptotics is going to be determined by a close neighbourhood of 0, so we take the expansion of $\cosh u$ at 0:

$$\begin{aligned} I &= \int_{-\infty}^{\infty} e^{-x} e^{-xu^2/2} e^{-x(u^4/4!+u^6/6!)} \\ &= e^{-x} \int_{-\infty}^{\infty} e^{xu^2/2}(1 - \frac{xu^4}{4} - \frac{xu^6}{6!} - \frac{xu^8}{8!} + \cdots) \\ &= e^{-x}(\sqrt{\frac{2\pi}{x}} + \frac{-x\sqrt{\pi}}{2!2^4(x/2)^{5/2}} + \cdots) \\ &= e^{-x}\sqrt{\frac{2\pi}{x}}(1 - \frac{1}{8x} + \cdots) \end{aligned}$$

Where we have used a standard result:

$$\int_{-\infty}^{\infty} t^{2n} e^{-At^2} \mathrm{d}t = \frac{(2n)!}{n!2^{2n}} \frac{[\sqrt{\pi}}{A^{n+1/2}}$$

Example.

$$\begin{aligned} I_x &= 2\int_0^{\infty} e^{-x\cosh u} \mathrm{d}u \\ &= 2\int_1^{\infty} e^{-xt} \frac{\mathrm{d}t}{\sqrt{t^2-1}} \\ &= s\int_0^{\infty} \frac{e^{-x}e^{-xs}}{\sqrt{s^2+2s}} \mathrm{d}s \\ &= \sqrt{2}\int_0^{\infty} e^{-x}e^{-xs}s^{-\frac{1}{2}}(1+\frac{s}{2})^{-1/2}\mathrm{d}s \\ &= \sqrt{2}e^{-x}\int_0^{\infty} e^{-xs}s^{-1/2}(1+(-\frac{1}{2})\frac{s}{2}+\frac{3}{8}\left(\frac{s}{2}\right)^2+\cdots)\mathrm{d}s \\ &= \sqrt{2}e^{-x}(x^{-1/2}\Gamma(1/2)+(-\frac{1}{4})x^{-3/2}\Gamma(3/2)) \\ &= e^{-x}(\sqrt{\frac{2\pi}{x}} - \frac{1}{8}\sqrt{\frac{2\pi}{x}}\frac{1}{x} + \cdots) \end{aligned}$$

And we can do this similarly as above.

Lemma (Watson's Lemma).

$$I(x) = \int_0^{\infty} e^{-xt} t^a g(t) \mathrm{d}t \sim \sum_{j=0}^{\infty} \frac{\alpha_j \Gamma(a+rj+1)}{x^{a+rj+1}}$$

where $a > -1$, $|g(t)| \le Ke^{tb}$, $g(t) \sim \sum \alpha_j t^{rj}$.

Proof.

$$|I(x) - \sum \frac{\alpha_j \Gamma(a+rj+1)}{x^{a+rj+1}}| = |\int_0^\infty e^{-xt}(t^a g(t) - \sum_0^N \alpha_j t^{a+rj})\mathrm{d}t|$$
$$= I + II$$

where we divided the integral at $t = R$, where for all $t < R$:

$$|g(t) - \sum_0^n \alpha_j t^{rj}| = |\alpha_{N+1} t^{r(N+1)} + o(t^{r(n+1)})| \leq C t^{r(n+1)}$$

Now we look at these integrals separately:

$$I = |\int_0^R e^{-xt}(t^a g(t) - \sum \alpha_j t^{a+rj})\mathrm{d}t| \leq C \int_0^R e^{-xt} t^{r(N+1)+a}\mathrm{d}t$$
$$\leq \frac{C\Gamma(a + r(N+1) + 1)}{x^{a+r(N+1)+1}} = o(x^{-(a+rN+1)})$$

where we have used that:

$$\int_0^\infty e^{-xt} t^{a+rj} = \frac{\Gamma(a+rj+1)}{c^{a+rj+1}}$$

Now we look at the second term:

$$II = |\int_R^\infty e^{-xt}(t^a g(t) - \sum_0^N \alpha_j t^{a+rj})\mathrm{d}t|$$
$$\leq K \int_R^\infty t^a e^{-xt+tb}\mathrm{d}t + \sum_0^N |\alpha_j| \int_R^\infty e^{-xt} t^{a+rj}\mathrm{d}t$$

Both terms are exponentially small, so we are done. □

Example. Find largest terms in the asymptotic expansion of:

$$I(x) = \int_0^{3\pi/2} e^{x(\sin t)^2}\mathrm{d}t$$

We expect asymptotics to arise from:

$$\int_{\pi/2-\epsilon}^{\pi/2+\epsilon} e^{x(\sin t)^2}\mathrm{d}t \qquad \int_{3\pi/2-\epsilon}^{3\pi/2} e^{x(\sin t)^2}\mathrm{d}t$$

Then we first look at:

$$\int_{3\pi/2-\epsilon}^{3\pi/2} e^{x(\sin t)^2}\mathrm{d}t = e^x \int_{3\pi/2-\epsilon}^{3\pi/2} e^{x((\sin t)^2-1)}\mathrm{d}t = e^x \int_0^{\delta(\epsilon)} \frac{e^{-xu}}{2\sqrt{u(1-u)}}$$

where $u = 1 - (\sin t)^2$. Then:

$$e^x \int_0^{\delta(\epsilon)} \frac{e^{-xu}}{2\sqrt{u(1-u)}} = \frac{e^x}{2} \int_0^{\delta(\epsilon)} e^{-xu} u^{-1/2}(1+u/2+\cdots) = \frac{e^x}{2}\sqrt{\frac{\pi}{x}}(1+\frac{1}{4x}+\cdots)$$

Then by symmetry, the contribution from the $\pi/2$ one should be two times the contribution at $\frac{3\pi}{2}$, so then we add these up to get the final answer:

$$I(x) \sim \frac{3e^x}{2}\sqrt{\frac{\pi}{x}}(1 + \frac{1}{4x} + \cdots)$$

3.0.1 Moving Critical Points

Now let's look at the following integral:

$$I(k) = \int_0^\infty e^{-\frac{1}{t}} e^{-kt} \mathrm{d}t \qquad k \to \infty$$

If we look at the maximum of $-(kt + \frac{1}{t})$, it occurs when $t = \frac{1}{\sqrt{k}}$. But this depends on k! We need to fix this. And we do so by letting $t = \frac{s}{\sqrt{k}}$. Then:

$$I(k) = \frac{1}{\sqrt{k}} \int_0^\infty e^{-\sqrt{k}(s+\frac{1}{s})} \mathrm{d}s$$

And we are back with a normal Laplace integral.

4 Fourier Integrals

Now we want to study:

$$I(k) = \int_a^b f(t) e^{ik\phi(t)} \mathrm{d}t k \to \infty$$

Wait...Isn't this the same? The i makes all the difference. Intuitively when k gets large, this oscillates more and more frequently, and we should have more cancellation between the positive and negative parts. Riemann and Lebesgue formalized it:

Lemma (Riemann-Lebesgue Lemma)**.** For $f(t) \in L^1$, meaning $\sup|f| < \infty$, we have:

$$\int_a^b f(t) e^{ik\phi(t)} \mathrm{d}t \to 0 \quad \text{as} \quad k \to \infty$$

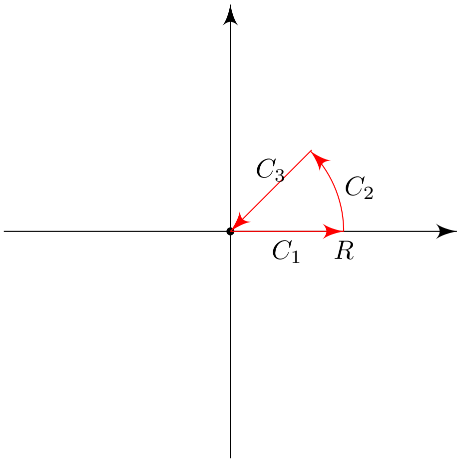
If $\phi(t)$ is monotonic, then taking $s = \phi(t)$ in a change of variables would result in the form above. Moreover, we have the following:

$$I(k) = \int_a^b f(t) e^{ikt} \mathrm{d}t \sim \sum_{m=0}^{N} \frac{(-1)^m}{(ik)^{m+1}} \left[e^{(m)}(b) e^{ikb} - f^{(m)}(a) e^{ika} \right] \quad k \to \infty$$

Given that $f(t)$ has $N+1$ continuous derivatives and has only finitely many discontinuities in its $N+2$nd derivative. Now we introduce a useful integral:

$$\int_0^\infty t^\gamma e^{ivt^p} \mathrm{d}t = \left(\frac{1}{|v|}\right)^{\frac{\gamma+1}{p}} \Gamma(\frac{\gamma+1}{p}) e^{\frac{i\pi}{s}(\gamma+1)} \frac{sgn(v)}{p} \quad p > 0, -1 < \gamma < 0, v \in \mathbb{R}$$

This integral is done by considering the following contour:



Taking $R \to \infty$, the integral on C_1 is what we want, the one on C_2 vanishes by Jordan's lemma, and parametrizing the third contour by $z = xe^{\frac{i\pi}{2p}}$, we can achieve the result above by noting that the sum of these integrals is 0 by Cauchy's theorem.

4.1 Method of Stationary Phase

What if $\phi'(t) = 0$ somewhere? Then, similar to Laplacian integrals, assume $\phi'(t)$ vanishes at a single interior point $a < c < b$ and has a non-vanishing second derivative. Then:

$$I(k) \sim \int_{c-\epsilon}^{c+\epsilon} f(c)e^{ik[\phi(c)+\frac{(t-c)^2}{2}\phi''(c)]}\mathrm{d}t \quad k \to \infty$$

Let $v = sgn(\phi''(c))$ and we change variables:

$$v\tau^2 = (t-c)^2\frac{\phi''(c)k}{2}$$

Then we have:

$$I(K) \sim \sqrt{\frac{2}{k|\phi''(c)|}}e^{ik\phi(c)}\int_{-\infty}^{\infty} e^{iv\tau^2}\mathrm{d}\tau = \sqrt{\frac{2\pi}{k|\phi''(c)|}}e^{ik\phi(c)}e^{\frac{i\pi}{4}sgn(\phi''(c))}$$

As we know the rest of the terms are of $O(\frac{1}{k})$ by integration by parts, this is the leading order contribution. Now we can extend this result:

Proposition.

$$I(K) = \int_a^b f(t)e^{ik\phi(t)}\mathrm{d}t, a < b < \infty$$

And we assume that:

(i) In the interval (a, b): we have that $\phi(t)$ and $\phi'(t)$ are continuous and $\phi'(t) > 0$ while $\phi''(t), f'(t)$ are only disctoninuous at finitely many points.

(ii) As $t \to a^+$, we have $\phi(t) - \phi(a) \sim (t-a)^\mu \Phi$, $f(t) \sim (t-a)^{\lambda-1}F$, with $\lambda < \mu$, and Φ, F, μ, λ positive constants. This ensures $\phi(t)$ is twice differentiable, ans $f(t)$ differentiable.

(iii) $\int_x^b |\frac{f(t)}{\phi'(t)}|\mathrm{d}t$ is finite for all $x \in (a, b)$.

(iv) As $t \to b_-$, we have that $\frac{f(t)}{\phi'(t)}$ tends to a finite limit.

Then we have:

$$I(k) \sim e^{\frac{i\pi\lambda}{2\mu}} \frac{F}{\mu} \Gamma(\frac{\lambda}{\mu}) \frac{e^{ik\phi(a)}}{(\Phi k)^{\frac{\lambda}{\mu}}} \quad k \to \infty$$

We would omit the proof here but it is similar as above.

Remark. What are we doing intuitively? If we have an i, we have oscillations. Oscillations cancel out as long as we don't have a stationary point of ϕ (in which the oscillation is much slower there), in which we do not have anything to cancel it out.

5 Method of Steepest Descent

We would now consider:

$$I(k) = \int_C f(z) e^{k\phi(z)} \mathrm{d}z$$

where $\phi(z)$ is now a function in the complex plane and C its contour. So basically we are now going to combine the stationary phase with Laplace's method. We call this *Method of Steepest Descent.* For every contour, as long as we don't move over a pole (even if we do we have the residue theorem to cover us), the integral is the same, so if we write:

$$\phi(z) = u(x, y) + iv(x, y)$$

where u, v are real functions, then if we deform the curve so that $v(x, y) = v(x_0, y_0)$ on the curve, then we can have:

$$I(k) = e^{ikv(x_0,y_0)} \int_C f(z) e^{ku(x,y)}$$

And we can use Laplace's method to help us from this point. These constant v curves are the curves where the change of $u(x, y)$ are maximized, or the steepest curves by definition, as $|\delta\phi|^2 = |\delta u|^2 + |\delta v|^2$, and $|\delta u|$ is maximized when $|\delta v|$ is minimized.

Now to use Laplace's method from this point, we know that we only care about the contour near a maximum point of f, which is when $f'(z) = 0$. Now assuming a non-vanishing second derivative:

$$f(z) = f(z_0) + \frac{1}{2}(z - z_0)^2 f''(z_0) + \cdots$$

Write $f''(z_0) = \rho e^{i\alpha}$ where we make it the expression unique by requiring $\rho > 0$ and $\alpha \in [0, 2\pi)$. When the contour is close to this point, we can assume it is a straight line and claim that it cacn be parametrized by $z - z_0 = re^{i\theta}$, $r > 0$. Then we substitute and have:

$$f(z) - f(z_0) = \frac{1}{2} r^2 \rho e^{i(2\theta+\alpha)}$$

From the discussion above, we want the imaginary part of this to be 0 (so that it is not changing), and the real part of this to be negative (so that $f(z_0)$ is a maximum). This gives:

$$\sin(2\theta + \alpha) = 0 \qquad \cos(2\theta + \alpha) < 0$$

These equations forces $\cos(2\theta + \alpha) = -1$, and we have:

$$\theta = -\frac{\alpha}{2} + \frac{\pi}{2}, \quad -\frac{\alpha}{2} + \frac{3\pi}{2}$$

So these are the directions that the contour should go through to get the Laplacian maximum and remove the changing imaginary part. Let's do an example to make this clear:

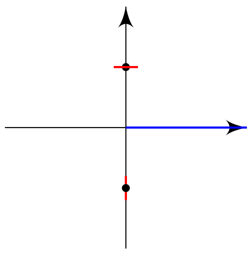
Example. We want to integrate the following:

$$I(x) = \int_0^\infty e^{i(\frac{1}{3}t^3+t)}\mathrm{d}t$$

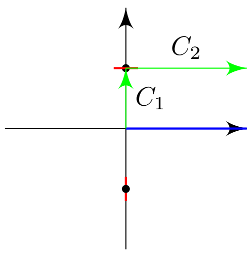
Before you shout at me and say this is the form of a stationary phase, I concede that it would be easier using SP but this would be a good and short example for method of steepest descent.

Now $f(z) = i(\frac{1}{3}z^3 + z)$, so its maximums occurs at $\pm i$, with $f''(z) = -\mp 2$. For $z_0 = i$, this gives $f''(z) = 2e^{i\pi}$, so the directions we want are $0, \pi$.

For $z_0 = -i$, this gives $f''(z) = 2e^{i\dot{0}}$, so the directions we want are $\frac{\pi}{2}, \frac{3\pi}{2}$. Then we can plot these on the complex plane and see what happens:



Now we want out blue contour to go through the red arrows. But how? We know that this graph has no poles so we can move these stuff around, but we start at 0. How do we pass these points? To do this, we must also consider the range in which this integral converges, which is, by using Jordan's lemma, between $\pm\tan^{-1}(\frac{1}{\sqrt{3}})$. So we can't go too far away. A sensible idea would be to go like:



And from there we know that the biggest contribution from C_2 is at $z = i$ (so we expand there) *BUT* we also need to consider the contribution from C_1 here as we did not go through the whole "maximum" curve, and we do that by parametrizing C_1 using $z - ix$ and noting that it becomes a standard Laplacian integral. The integral at C_2 can be written as:

$$I_{C_2} = \int_i^{i+\epsilon e^{i\theta}} e^{(-\frac{2}{3}-(z-i)^2)\mathrm{d}z}$$

Which is easy to take it from here as $\theta = 0$ and we can take $z' = z - i$.

Note. Wait, wait wait. Here it is easy because we are integrating in the real direction. But what if we are not? If we are not, we take $z - i = z'e^{i\theta}$ and since we *defined* $f''(z_0) = \rho e^{\imath\alpha}$, the $e^{i2\theta}$ term that comes out (as the exponent becomes $f(z_0) - z'^2 e^{2i\theta} e^{i\alpha}$) after the substitution cancels out with $e^{i\alpha}$ to give -1 in the exponent as this is how we defined θ in the first place. Smart!

6 Asymptotics for Differential Equations

6.1 Liouville-Green Method

We would like to solve:

$$u'' + \lambda^2 u(x) y = 0$$

Where λ is large. Why? This is useful in quantum mechanics for perturbation theory, and in waves for transient solutions, among others. But we digress. Employing Magic^{TM}, we look for solutions of the form:

$$y(x) = e^{\lambda S(x)} \left(z_0(x) + \frac{z_1(x)}{\lambda} + O(\frac{1}{\lambda^2}) \right)$$

We substitute this back into the original equation and demand that the $O(\lambda)$ and the $O(\lambda^2)$ terms vanish:

$$\begin{cases} S'(x)^2 + u(x) & = 0 \\ z_0 S''(x) + 2S'(x) z_0 & = 0 \end{cases}$$

Then the first equation gives:

$$S(x) = \pm i \int^x \sqrt{u} \mathrm{d}t$$

And substituting that into the second one gives you:

$$\mathrm{d}(\ln z_0) = -\frac{1}{2} \mathrm{d}(\ln(S'))$$

So $z_0 = bu^{-\frac{1}{4}}$. Then we have:

$$y(x) \sim \frac{1}{u^{\frac{1}{4}}} \left(c_1 e^{i\lambda \int^x u^{\frac{1}{2}} \mathrm{d}t} + c_2 e^{-i\lambda \int^x u^{\frac{1}{2}} \mathrm{d}t} \right)$$

Remark. If we have a differential equation $w'' + a_1(x) + a_0 w = 0$, then we can map this to the form required by:

$$w = y e^{-\frac{1}{2} \int^x a_1(t) \mathrm{d}t}$$

6.2 WKBJ Method

Wait, didn't we finish everything already? No! When u vanishes, we don't have a solution, but we can have a different method. Expand u about x_0 where $u(x_0) = 0$, then:

$$u(x) \approx u'(x_0)(x - x_0)$$

To leading order. We then have:

$$y'' + \lambda^2 u'(x_0)(x - x_0)y = 0$$

We do a variable change of $t = x - x_0$ to see that:

$$y'' = \lambda^2 u'(x_0)ty$$

Now take $t = \alpha z$ to get:

$$y''(z) = \lambda^2 u'(x_0)\alpha^3 zy$$

Then we choose $\alpha = (u'(x_0))^{-\frac{1}{3}}\lambda^{-\frac{2}{3}}$. The equation then becomes the Airy equation, in which we know it has the solution:

$$y(x) \sim C_1 A_i(k^{1/3}\lambda^{2/3}(x - x_0)) + C_2 B_i(k^{1/3}\lambda^{2/3}(x - x_0))$$

But for $x \neq x_0$, we also have the solution, assuming $u(x) < 0$ for $x > x_0$:

$$y_R(x) \sim \frac{1}{|u(x)|^{1/4}} \left(C_1^R e^{\lambda \int^x (-u)^{1/2}} \mathrm{d}t + C_2^R e^{-\lambda \int^x (-u)^{1/2}} \mathrm{d}t\right)$$

This is valid for $x \to x_0^+$. For $x \to x_0^-$:

$$y_L(x) \sim \frac{1}{|u(x)|^{1/4}} \left(C_1^L e^{i\lambda \int^x u^{1/2}} \mathrm{d}t + C_2^L e^{-i\lambda \int^x u^{1/2}} \mathrm{d}t\right)$$

And we employ the asymptotic expansion that:

$$A_i(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}} \quad B_i(x) \sim \frac{e^{\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}}, x \to \infty$$

And

$$A_i(x) \sim \frac{\sin(\frac{2}{3}|x|^{3/2} + \frac{\pi}{4})}{\sqrt{\pi}|x|^{1/4}} \quad B_i(x)\frac{\cos(\frac{2}{3}|x|^{3/2} + \frac{\pi}{4})}{\sqrt{\pi}|x|^{1/4}}, x \to -\infty$$

After expanding these into powers, we can match up the coefficients $C_1, C_2, C_1^R, C_2^R, C_1^L, C_2^L$ in their respective powers. For example, we have:

$$C_1^R = \frac{C_1}{2\sqrt{\pi}} \left(\frac{u'(x_0)}{\lambda}\right)^{1/6} \quad C_2^R = \frac{C_2}{2\sqrt{\pi}} \left(\frac{u'(x_0)}{\lambda}\right)^{1/6}$$

Now we are actually done.